ECE 5510 Fall 2009: Homework 10 Solutions

1. Y&G 11.6.1: See additional pages.

2. (a) $S_X(\phi) = \text{DTFT}\left\{(\sqrt{1/2})^{|k|}\right\} = \frac{1/2}{3/2-\sqrt{2}\cos(2\pi\phi)}$.

   (b) $H(\phi) = \text{DTFT}\left\{(\sqrt{1/2})^{n}u[n]\right\} = \frac{1}{1-\sqrt{1/2}e^{-j2\pi\phi}}$

   (c) Note that

$$|H(\phi)|^2 = \frac{1}{1-\sqrt{1/2}e^{-j2\pi\phi}}\frac{1}{1-\sqrt{1/2}e^{+j2\pi\phi}} = \frac{1}{3/2-\sqrt{2}\cos(2\pi\phi)}$$

   So,

$$S_Y(\phi) = S_X(\phi)|H(\phi)|^2 = \frac{1/2}{[3/2-\sqrt{2}\cos(2\pi\phi)]^2}$$

3. For parts (b) and (c) of this problem, you could either (1) take the DTFT of the difference equation, to find the frequency response of the transfer function, and then use that result to find $S_Y(\phi)$ and then $R_Y[k]$, or (2) generate $R_Y[k]$ without the DTFT, using the definition of the autocorrelation function.

   (a) Taking the DTFT of both sides of $Y_n = a_1 Y_{n-1} + X_n$, and then solving,

$$\begin{aligned} Y(\phi) &= a_1 Y(\phi) e^{-j2\pi\phi} + X(\phi) \\ Y(\phi)\left[1 - a_1 e^{-j2\pi\phi}\right] &= X(\phi) \\ H(\phi) = \frac{Y(\phi)}{X(\phi)} &= \frac{1}{1-a_1 e^{-j2\pi\phi}} \end{aligned} \tag{1}$$

   (b) I wrote parts (b) and (c) out of order. Using the result of part (c),

$$\begin{aligned} R_Y(k) &= \text{DTFT}^{-1}\left\{\frac{\sigma_X^2}{1+a_1^2-2a_1\cos(2\pi\phi)}\right\} \\ R_Y(k) &= \frac{\sigma_X^2}{1-a_1^2}a_1^{|k|} \end{aligned} \tag{2}$$

   (c) Note $R_X(k) = \sigma_X^2\delta[k]$. Thus $S_X(\phi) = \sigma_X^2$. So

$$\begin{aligned} S_Y(\phi) &= S_X(\phi)|H(\phi)|^2 = \sigma_X^2 \frac{1}{1-a_1e^{-j2\pi\phi}}\frac{1}{1-a_1e^{+j2\pi\phi}} \\ &= \frac{\sigma_X^2}{1+a_1^2-2a_1\cos(2\pi\phi)} \end{aligned} \tag{3}$$

(d) This note is an alternate way to find the solution to (b), purely in the time domain. Starting at time $n+k$ the output equation is,

$$Y_{n+k} = a_1 Y_{n+k-1} + X_{n+k}$$

Then plugging in recursively for Y_{n+k-1},

$$\begin{aligned} Y_{n+k} &= a_1(a_1 Y_{n+k-2} + X_{n+k-1}) + X_{n+k} = a_1^2 Y_{n+k-2} + a_1 X_{n+k-1} + X_{n+k} \\ Y_{n+k} &= a_1^2(a_1 Y_{n+k-3} + X_{n+k-2}) + a_1 X_{n+k-1} + X_{n+k} \\ &= a_1^3 Y_{n+k-3} + a_1^2 X_{n+k-2} + a_1 X_{n+k-1} + X_{n+k} \end{aligned}$$

We can see that after k iterations of this, we'd have

$$\begin{aligned} Y_{n+k} &= a_1^k Y_n + a_1^{k-1} X_{n+1} + \cdots + a_1^2 X_{n+k-2} + a_1 X_{n+k-1} + X_{n+k} \\ &= a_1^k Y_n + \sum_{i=1}^{k} a_1^{k-i} X_{n+i} \end{aligned}$$

Then, we find $R_Y[k]$. Assuming that $k > 0$,

$$R_Y[k] = E[Y_n Y_{n+k}] = E\left[a_1^k Y_n^2 + \sum_{i=1}^{k} a_1^{k-i} X_{n+i} Y_n\right]$$

Because of the independence of X_{n+i} and Y_n,

$$R_Y[k] = a_1^k E\left[Y_n^2\right] + \sum_{i=1}^{k} a_1^{k-i} E[X_{n+i}]\, E[Y_n]$$

Since we know that X_n is a zero-mean r.p., the term in the sum disappears.

$$R_Y[k] = a_1^k E\left[Y_n^2\right] \tag{4}$$

To find the mean (and variance), we can go back to the original expression for Y_n and take the expected value (and the variance) of both sides,

$$\begin{aligned} E[Y_{n+k}] &= a_1 E[Y_{n+k-1}] + E[X_{n+k}] \\ \text{Var}[Y_{n+k}] &= a_1 \text{Var}[Y_{n+k-1}] + \text{Var}[X_{n+k}] \end{aligned}$$

Since Y is WSS, the mean and variance are both constant,

$$\begin{aligned} \mu_Y(1-a_1) &= E[X_{n+k}] = 0 \quad \text{thus} \quad \mu_Y = 0 \\ \sigma_Y^2 &= \frac{1}{1-a_1}\sigma_X^2 \end{aligned}$$

Plugging these back into (4),

$$R_Y[k] = \frac{\sigma_X^2}{1-a_1} a_1^k$$

Now, for the $k < 0$ case, note that

$$Y_n = a_1^{-k} Y_{n+k} + \sum_{i=1}^{k} a_1^{k-i} X_{n+i+k}$$

So for the $k < 0$ case, similar analysis to that above results in $R_Y[k] = \frac{\sigma_X^2}{1-a_1} a_1^{-k}$. So overall, for all k, we have

$$R_Y[k] = \frac{\sigma_X^2}{1-a_1} a_1^{|k|}$$

This agrees with our frequency domain solution written above.

4. From the definition of the DTFT,

$$S_X(\phi) = 1e^{-j2\pi(-1)\phi} + 2e^{-j2\pi(0)\phi} + 1e^{-j2\pi(1)\phi} = 2 + 2\cos(2\pi\phi)$$

Since $H(\phi) = 10$,

$$S_Y(\phi) = |H(\phi)|^2 S_X(\phi) = 200[1 + \cos(2\pi\phi)]$$

5. (a) H(f) is given by

$$H(f) = \frac{0.1}{0.1 + j2\pi f}$$

Thus

$$S_X(f) = S_N(f)|H(f)|^2 = \frac{0.01}{0.01 + (2\pi f)^2}$$

(b) Since $N(t)$ has zero-mean, and $h(t)$ is an LTI filter, $X(t)$ is zero mean and $C_X(\tau) = R_X(\tau)$,

$$R_X(\tau) = \mathfrak{F}^{-1}\left\{\frac{2}{2}\frac{0.01}{0.01 + (2\pi f)^2}\right\} = 0.05e^{-0.1|\tau|}$$

Thus the variance is $\sigma_{X(t)}^2 = R_X(0) = 0.05$.

(c) Since $N(t)$ is Gaussian and $h(t)$ is an LTI filter, $X(0.2)$ is Gaussian. Thus

$$P[X(0.2) > 0.1] = P\left[\frac{X(0.2)}{\sqrt{0.05}} > \frac{0.1}{\sqrt{0.05}}\right] = 1 - \Phi\left(\frac{0.1}{\sqrt{0.05}}\right) \approx 0.3274$$

(d) Because of the Gaussian input and the LTI filter, $X(1), X(6)$ are jointly Gaussian, both with zero mean and variance 0.05. The covariance is based on a τ of $6 - 1 = 5$, so $C_X(5) = 0.05e^{-0.5}$, for a correlation coefficient of $\rho = e^{-0.5}$. As a result, we can write the joint pdf,

$$f_{X(1),X(6)}(x_1, x_6) = \frac{1}{0.1\pi\sqrt{1 - e^{-1}}} \exp\left\{-\frac{1}{0.1(1 - e^{-1})}\left[x_1^2 - 2\frac{e^{-0.5}}{0.05}x_1 x_6 + x_6^2\right]\right\}$$

6. Y&G 12.1.1: See additional pages.

7. Y&G 12.2.2: Since $\mathbf{P}(1) = \mathbf{P}$ as given above, we can use matrix multiplication to find $\mathbf{P}(2)$, $\mathbf{P}(3)$ and $\mathbf{P}(4)$:

$$\mathbf{P}(2) = \mathbf{P}(1) \cdot \mathbf{P}(1) = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{bmatrix} \quad \mathbf{P}(3) = \mathbf{P}(2) \cdot \mathbf{P}(1) = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ \frac{7}{16} & \frac{7}{16} & \frac{1}{8} \end{bmatrix}$$

$$\mathbf{P}(4) = \mathbf{P}(3) \cdot \mathbf{P}(1) = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ \frac{15}{32} & \frac{15}{32} & \frac{1}{16} \end{bmatrix} \tag{5}$$

It is clear that the first two rows stay identical each time. In the 3rd row, the first two elements get $1/2^{n+1}$ closer to $1/2$ each time, while the last element is $1/2^n$:

$$\mathbf{P}(n) = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ \frac{1}{2} - \frac{1}{2^{n+1}} & \frac{1}{2} - \frac{1}{2^{n+1}} & \frac{1}{2^n} \end{bmatrix} \tag{6}$$

In Matlab, the following code:

```
P = [0.5, 0.5, 0; 0.5, 0.5, 0; 0.25, 0.25, 0.5];
for n=2:4,
    error = P^n - [0.5, 0.5, 0; 0.5, 0.5, 0; ...
                  0.5-(0.5)^(n+1), 0.5-(0.5)^(n+1), (0.5)^n]
end
```

returns 'error' matrices of all zeros, indicating that the model of (6) and the directly calculated $\mathbf{P}^n$ are identical for $n = 2, 3, 4$.

Problem 11.6.1 Solution

Since the random sequence X_n has autocorrelation function

$$R_X[k] = \delta_k + (0.1)^{|k|}, \tag{1}$$

We can find the PSD directly from Table 11.2 with $0.1^{|k|}$ corresponding to $a^{|k|}$. The table yields

$$S_X(\phi) = 1 + \frac{1-(0.1)^2}{1+(0.1)^2 - 2(0.1)\cos 2\pi\phi} = \frac{2-0.2\cos 2\pi\phi}{1.01-0.2\cos 2\pi\phi}. \tag{2}$$

Problem 11.7.1 Solution

First we show that $S_{YX}(f) = S_{XY}(-f)$. From the definition of the cross spectral density,

$$S_{YX}(f) = \int_{-\infty}^{\infty} R_{YX}(\tau) e^{-j2\pi f\tau}\, d\tau \tag{1}$$

Making the subsitution $\tau' = -\tau$ yields

$$S_{YX}(f) = \int_{-\infty}^{\infty} R_{YX}(-\tau') e^{j2\pi f\tau'}\, d\tau' \tag{2}$$

By Theorem 10.14, $R_{YX}(-\tau') = R_{XY}(\tau')$. This implies

$$S_{YX}(f) = \int_{-\infty}^{\infty} R_{XY}(\tau') e^{-j2\pi(-f)\tau'}\, d\tau' = S_{XY}(-f) \tag{3}$$

To complete the problem, we need to show that $S_{XY}(-f) = [S_{XY}(f)]^*$. First we note that since $R_{XY}(\tau)$ is real valued, $[R_{XY}(\tau)]^* = R_{XY}(\tau)$. This implies

$$[S_{XY}(f)]^* = \int_{-\infty}^{\infty} [R_{XY}(\tau)]^* [e^{-j2\pi f\tau}]^*\, d\tau \tag{4}$$

$$= \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j2\pi(-f)\tau}\, d\tau \tag{5}$$

$$= S_{XY}(-f) \tag{6}$$

Problem 11.8.1 Solution

Let $a = 1/RC$. The solution to this problem parallels Example 11.22.

(a) From Table 11.1, we observe that

$$S_X(f) = \frac{2\cdot 10^4}{(2\pi f)^2 + 10^4} \qquad H(f) = \frac{1}{a + j2\pi f} \tag{1}$$

By Theorem 11.16,

$$S_Y(f) = |H(f)|^2 S_X(f) = \frac{2\cdot 10^4}{[(2\pi f)^2 + a^2][(2\pi f)^2 + 10^4]} \tag{2}$$

To find $R_Y(\tau)$, we use a form of partial fractions expansion to write

$$S_Y(f) = \frac{A}{(2\pi f)^2 + a^2} + \frac{B}{(2\pi f)^2 + 10^4} \tag{3}$$

Problem Solutions – Chapter 12

Problem 12.1.1 Solution

From the given Markov chain, the state transition matrix is

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0.25 & 0.25 & 0.5 \end{bmatrix} \tag{1}$$

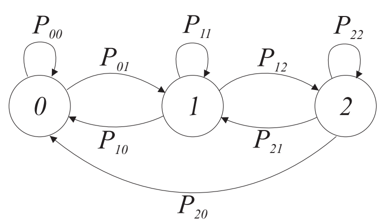
Problem 12.1.2 Solution

This problem is very straightforward if we keep in mind that P_{ij} is the probability that we transition from state i to state j. From Example 12.1, the state transition matrix is

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \tag{1}$$

Problem 12.1.3 Solution

In addition to the normal OFF and ON states for packetized voice, we add state 2, the "mini-OFF" state. The Markov chain is

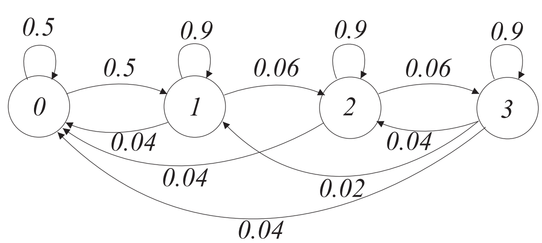


The only difference between this chain and an arbitrary 3 state chain is that transitions from 0, the OFF state, to state 2, the mini-OFF state, are not allowed. From the problem statement, the corresponding Markov chain is

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 0.999929 & 0.000071 & 0 \\ 0.000100 & 0.899900 & 0.1 \\ 0.000100 & 0.699900 & 0.3 \end{bmatrix}. \tag{1}$$

Problem 12.1.4 Solution

Based on the problem statement, the state of the wireless LAN is given by the following Markov chain:



The Markov chain has state transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.04 & 0.9 & 0.06 & 0 \\ 0.04 & 0 & 0.9 & 0.06 \\ 0.04 & 0.02 & 0.04 & 0.9 \end{bmatrix}. \tag{1}$$