# ECE 5510 Fall 2009: Homework 1 Solutions

1. Y&G 1.2.1:

   (a) An outcome specifies whether the fax is high (h), medium (m), or low (l) speed, and whether the fax has two (t) pages or four (f) pages. The sample space is $S = \{ht, hf, mt, mf, lt, lf\}$.

   (b) The event that the fax is medium speed is $A_1 = \{mt, mf\}$.

   (c) The event that a fax has two pages is $A_2 = \{ht, mt, lt\}$.

   (d) The event that a fax is either high speed or low speed is $A_3 = \{ht, hf, lt, lf\}$.

   (e) Since $A_1 \cap A_2 = \{mt\}$ and is not empty, $A_1$, $A_2$, and $A_3$ are not mutually exclusive.

   (f) Since $A_1 \cup A_2 \cup A_3 = \{ht, hf, mt, mf, lt, lf\} = S$, the collection $A_1$, $A_2$, $A_3$ is collectively exhaustive.

2. If $A \subset B$, then $B^c \subset A^c$. Proof: The definition of $A \subset B$ is: if $x \in A$, then $x \in B$. Then, the contrapositive is that, if $x \notin B$, then $x$ cannot be in set $A$. This implies that if $x \in B^c$, then $x \in A^c$, which is equivalent to $B^c \subset A^c$. Since I didn't ask specifically for a proof, other valid justifications may exist.

3. Y&G 1.4.2:

   (a) The probability that a call is billed for more than 3 minutes is

$$\begin{aligned} P[L] &= 1 - \text{P[3 or fewer billed minutes]} \\ &= 1 - P[B_1] - P[B_2] - P[B_3] \\ &= 1 - \alpha - \alpha(1-\alpha) - \alpha(1-\alpha)^2 \\ &= (1-\alpha)^3 = 0.57. \end{aligned}$$

   (b) The probability that a call will billed for 9 minutes or less is, using the fact that $\sum_{i=0}^{n} a^n = \frac{1-a^{n+1}}{1-a}$,

$$\begin{aligned} P\left[9 \text{ minutes or less}\right] &= \sum_{i=1}^{9} P\left[B_i\right] = \sum_{i=1}^{9} \alpha(1-\alpha)^{i-1} \\ &= \alpha \sum_{i=0}^{8} (1-\alpha)^i = \alpha \frac{1-(1-\alpha)^9}{\alpha} = 1 - (1-\alpha)^9 \qquad (1) \end{aligned}$$

   Since $0.57 = (1-\alpha)^3$, the answer is $1 - (0.57)^3 = 0.8148$.

4. We must prove that $P\left[E \oplus F\right] = P\left[E \cap F^c\right] + P\left[E^c \cap F\right]$. If we define $A = E \cap F^c$ and $B = F \cap E^c$, we can show that $A$ and $B$ are disjoint. To prove disjoint, we show that their intersection is the null set:

$$A \cap B = (E \cap F^c) \cap (F \cap E^c) = E \cap (F^c \cap F) \cap E^c = \emptyset$$

   since the intersection of a set and its complement is the null set, so this pair of events is disjoint. Finally, we can apply axiom 3 since the two sets ($A$ and $B$) are disjoint.

$$P\left[E \oplus F\right] = P\left[(E \cap F^c) \cup (F \cap E^c)\right] = P\left[E \cap F^c\right] + P\left[E^c \cap F\right]$$

5. Group the three events $E_1$, $E_2$, and $E_3$, into two groups, $E_1$ and $(E_2 \cup E_3)$, and repeatedly apply the law that $P[A \cup B] = P[A] + P[B] - P[A \cap B]$:

$$\begin{aligned}
P[E_1 \cup (E_2 \cup E_3)] &= P[E_1] + P[E_2 \cup E_3] - P[E_1 \cap (E_2 \cup E_3)] \\
&= P[E_1] + P[E_2] + P[E_3] - P[E_2 \cap E_3] - P[(E_1 \cap E_2) \cup (E_1 \cap E_3)] \\
&= P[E_1] + P[E_2] + P[E_3] - P[E_2 \cap E_3] - P[E_1 \cap E_2] - P[E_1 \cap E_3] \\
&\quad + P[(E_1 \cap E_2) \cap (E_1 \cap E_3)] \\
&= P[E_1] + P[E_2] + P[E_3] - P[E_2 \cap E_3] - P[E_1 \cap E_2] - P[E_1 \cap E_3] \\
&\quad + P[E_1 \cap E_2 \cap E_3].
\end{aligned}$$