ECE 5510 Fall 2009: Homework 2 Solutions

1. Y&G 1.5.4. See attached pages.
2. Y&G 1.6.4. See attached pages.
3. Y&G 1.7.4. See attached pages.
4. Y&G 1.7.10. See attached pages.
5. Since k balls can be selected from the $n+m$ balls, and order doesn't matter, there are $|S| = \binom{n+m}{k}$ possible outcomes in the sample space.

   (a) Let the event $A_r = \{r \text{ of the } k \text{ balls are red}\}$. Only when $r \leq k$ is the event A_r possible (otherwise it has zero probability). Similarly, it must be that $k - r \leq m$ and $r \leq n$. Since any combination is equally likely, we use the discrete uniform probability law to say that $P[A_r] = \frac{|A_r|}{|S|}$. Again, A_r consists of outcomes in which r red balls (and thus $k-r$ black balls) are chosen. Assuming that the above inequalities are satisfied, we obtain $|A_r|$ by counting each of the ways that r red balls are drawn from among the n red balls, $\binom{n}{r}$, and the ways that the remaining $k-r$ balls are drawn among the m black balls, $\binom{m}{k-r}$. Thus the total number of favorable outcomes $\binom{n}{r}\binom{m}{k-r}$.

   $$P[A_r] = \frac{|A_r|}{|S|} = \frac{\binom{n}{r}\binom{m}{k-r}}{\binom{n+m}{k}}$$

   (b) Here, note that $\{A_r\}$ for $r = 0, 1, \ldots, k$ form a partition of the sample space. This is because $A_{r_0} \cap A_{r_1} = \emptyset$ for any $r_0 \neq r_1$ since we can't have any outcome which has exactly r_0 red balls and exactly r_1 red balls in it. Further, there are no outcomes that don't have between 0 and k red balls in them. Since $\{A_r\}$ is a partition of S, $\bigcup_{r=0}^{k} A_r = S$ and thus

   $$\begin{aligned}\binom{n}{0}\binom{m}{k} + \binom{n}{1}\binom{m}{k-1} + \cdots + \binom{n}{k}\binom{m}{0} &= \sum_{r=0}^{k}\binom{n}{r}\binom{m}{k-r} = \sum_{r=0}^{k}|A_r| \\ &= \left|\bigcup_{r=0}^{k} A_r\right| = |S| = \binom{n+m}{k}\end{aligned}$$

   The first equality on the second line is due to the fact that $\{A_r\}$ are mutually exclusive.

6. Let the events be:

   - dieA: choose die A
   - dieB: choose die B
   - O_n: olive face on throw n
   - L_n: lavender face on throw n

   (a) The nth throw is no different from any throw of the die. Using the law of total probability,

   $$\begin{aligned}P[O_n] &= P[O_n \cap \text{dieA}] + P[O_n \cap \text{dieB}] \\ &= P[O_n|\text{dieA}]\,P[\text{dieA}] + P[O_n|\text{dieB}]\,P[\text{dieB}] \\ &= (5/6)(1/2) + (1/2)(1/2) = 2/3\end{aligned}$$

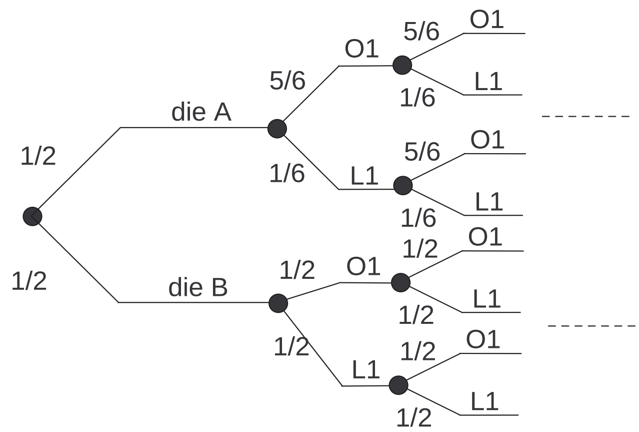


Figure 1: Tree.

(b) These two throws are NOT independent! They both depend on the outcome of the one coin flip.

$$\begin{aligned} P[O_n \cap O_{n+1}] &= P[O_n \cap O_{n+1} \cap \text{dieA}] + P[O_n \cap O_{n+1} \cap \text{dieB}] \\ &= P[O_n \cap O_{n+1} | \text{dieA}] P[\text{dieA}] + P[O_n \cap O_{n+1} | \text{dieB}] P[\text{dieB}] \\ &= (5/6)(5/6)(1/2) + (1/2)(1/2)(1/2) = 17/36 \end{aligned}$$

(c)

$$P[O_{n+1} | O_1 \cap \cdots \cap O_n] = \frac{P[O_{n+1} \cap O_n \cap \cdots \cap O_1]}{P[O_1 \cap \cdots \cap O_n]}$$

Extrapolating from (b), this is

$$\begin{aligned} P[O_{n+1} | O_1 \cap \cdots \cap O_n] &= \frac{(5/6)^{n+1}(1/2) + (1/2)^{n+1}(1/2)}{(5/6)^n(1/2) + (1/2)^n(1/2)} \\ &= \frac{(5/6) + (1/2)(3/5)^n}{1 + (3/5)^n} \end{aligned}$$

As n becomes large, $(3/5)^n \to 0$. Thus the numerator approaches 5/6 and the denominator approaches 1. So $P[O_{n+1} | O_1 \cap \cdots \cap O_n]$ approaches 5/6, indicating that die A is being used.

Problem 1.5.2 Solution

Let s_i denote the outcome that the roll is i. So, for $1 \le i \le 6$, $R_i = \{s_i\}$. Similarly, $G_j = \{s_{j+1}, \ldots, s_6\}$.

(a) Since $G_1 = \{s_2, s_3, s_4, s_5, s_6\}$ and all outcomes have probability 1/6, $P[G_1] = 5/6$. The event $R_3G_1 = \{s_3\}$ and $P[R_3G_1] = 1/6$ so that

$$P[R_3|G_1] = \frac{P[R_3G_1]}{P[G_1]} = \frac{1}{5}. \tag{1}$$

(b) The conditional probability that 6 is rolled given that the roll is greater than 3 is

$$P[R_6|G_3] = \frac{P[R_6G_3]}{P[G_3]} = \frac{P[s_6]}{P[s_4, s_5, s_6]} = \frac{1/6}{3/6}. \tag{2}$$

(c) The event E that the roll is even is $E = \{s_2, s_4, s_6\}$ and has probability 3/6. The joint probability of G_3 and E is

$$P[G_3E] = P[s_4, s_6] = 1/3. \tag{3}$$

The conditional probabilities of G_3 given E is

$$P[G_3|E] = \frac{P[G_3E]}{P[E]} = \frac{1/3}{1/2} = \frac{2}{3}. \tag{4}$$

(d) The conditional probability that the roll is even given that it's greater than 3 is

$$P[E|G_3] = \frac{P[EG_3]}{P[G_3]} = \frac{1/3}{1/2} = \frac{2}{3}. \tag{5}$$

Problem 1.5.3 Solution

Since the 2 of clubs is an even numbered card, $C_2 \subset E$ so that $P[C_2E] = P[C_2] = 1/3$. Since $P[E] = 2/3$,

$$P[C_2|E] = \frac{P[C_2E]}{P[E]} = \frac{1/3}{2/3} = 1/2. \tag{1}$$

The probability that an even numbered card is picked given that the 2 is picked is

$$P[E|C_2] = \frac{P[C_2E]}{P[C_2]} = \frac{1/3}{1/3} = 1. \tag{2}$$

Problem 1.5.4 Solution

Define D as the event that a pea plant has two dominant y genes. To find the conditional probability of D given the event Y, corresponding to a plant having yellow seeds, we look to evaluate

$$P[D|Y] = \frac{P[DY]}{P[Y]}. \tag{1}$$

Note that $P[DY]$ is just the probability of the genotype yy. From Problem 1.4.3, we found that with respect to the color of the peas, the genotypes yy, yg, gy, and gg were all equally likely. This implies

$$P[DY] = P[yy] = 1/4 \qquad P[Y] = P[yy, gy, yg] = 3/4. \tag{2}$$

Thus, the conditional probability can be expressed as

$$P[D|Y] = \frac{P[DY]}{P[Y]} = \frac{1/4}{3/4} = 1/3. \tag{3}$$

(c) Since C and D are independent,

$$P[C \cap D] = P[C]\,P[D] = 15/64. \tag{3}$$

The next few items are a little trickier. From Venn diagrams, we see

$$P[C \cap D^c] = P[C] - P[C \cap D] = 5/8 - 15/64 = 25/64. \tag{4}$$

It follows that

$$P[C \cup D^c] = P[C] + P[D^c] - P[C \cap D^c] \tag{5}$$
$$= 5/8 + (1 - 3/8) - 25/64 = 55/64. \tag{6}$$

Using DeMorgan's law, we have

$$P[C^c \cap D^c] = P[(C \cup D)^c] = 1 - P[C \cup D] = 15/64. \tag{7}$$

(d) Since $P[C^cD^c] = P[C^c]P[D^c]$, C^c and D^c are independent.

Problem 1.6.4 Solution

(a) Since $A \cap B = \emptyset$, $P[A \cap B] = 0$. To find $P[B]$, we can write

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] \tag{1}$$
$$5/8 = 3/8 + P[B] - 0. \tag{2}$$

Thus, $P[B] = 1/4$. Since A is a subset of B^c, $P[A \cap B^c] = P[A] = 3/8$. Furthermore, since A is a subset of B^c, $P[A \cup B^c] = P[B^c] = 3/4$.

(b) The events A and B are dependent because

$$P[AB] = 0 \neq 3/32 = P[A]\,P[B]. \tag{3}$$

(c) Since C and D are independent $P[CD] = P[C]P[D]$. So

$$P[D] = \frac{P[CD]}{P[C]} = \frac{1/3}{1/2} = 2/3. \tag{4}$$

In addition, $P[C \cap D^c] = P[C] - P[C \cap D] = 1/2 - 1/3 = 1/6$. To find $P[C^c \cap D^c]$, we first observe that

$$P[C \cup D] = P[C] + P[D] - P[C \cap D] = 1/2 + 2/3 - 1/3 = 5/6. \tag{5}$$

By De Morgan's Law, $C^c \cap D^c = (C \cup D)^c$. This implies

$$P[C^c \cap D^c] = P[(C \cup D)^c] = 1 - P[C \cup D] = 1/6. \tag{6}$$

Note that a second way to find $P[C^c \cap D^c]$ is to use the fact that if C and D are independent, then C^c and D^c are independent. Thus

$$P[C^c \cap D^c] = P[C^c]\,P[D^c] = (1 - P[C])(1 - P[D]) = 1/6. \tag{7}$$

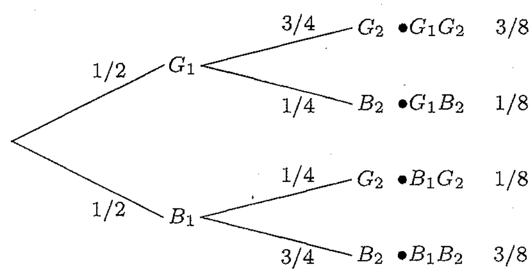
Finally, since C and D are independent events, $P[C|D] = P[C] = 1/2$.

(d) Note that we found $P[C \cup D] = 5/6$. We can also use the earlier results to show

$$P[C \cup D^c] = P[C] + P[D] - P[C \cap D^c] = 1/2 + (1 - 2/3) - 1/6 = 2/3. \tag{8}$$

(e) By Definition 1.7, events C and D^c are independent because

$$P[C \cap D^c] = 1/6 = (1/2)(1/3) = P[C]\,P[D^c]. \tag{9}$$

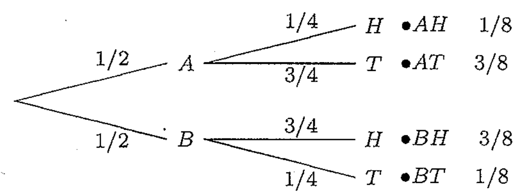


The game goes into overtime if exactly one free throw is made. This event has probability

$$P[O] = P[G_1B_2] + P[B_1G_2] = 1/8 + 1/8 = 1/4. \tag{1}$$

Problem 1.7.4 Solution

The tree for this experiment is



The probability that you guess correctly is

$$P[C] = P[AT] + P[BH] = 3/8 + 3/8 = 3/4. \tag{1}$$

Problem 1.7.5 Solution

The $P[-|H]$ is the probability that a person who has HIV tests negative for the disease. This is referred to as a false-negative result. The case where a person who does not have HIV but tests positive for the disease, is called a false-positive result and has probability $P[+|H^c]$. Since the test is correct 99% of the time,

$$P[-|H] = P[+|H^c] = 0.01. \tag{1}$$

Now the probability that a person who has tested positive for HIV actually has the disease is

$$P[H|+] = \frac{P[+,H]}{P[+]} = \frac{P[+,H]}{P[+,H] + P[+,H^c]}. \tag{2}$$

We can use Bayes' formula to evaluate these joint probabilities.

$$\begin{aligned} P[H|+] &= \frac{P[+|H]\,P[H]}{P[+|H]\,P[H] + P[+|H^c]\,P[H^c]} & (3)\\ &= \frac{(0.99)(0.0002)}{(0.99)(0.0002) + (0.01)(0.9998)} & (4)\\ &= 0.0194. & (5) \end{aligned}$$

Thus, even though the test is correct 99% of the time, the probability that a random person who tests positive actually has HIV is less than 0.02. The reason this probability is so low is that the a priori probability that a person has HIV is very small.

Problem 1.7.9 Solution

(a) We wish to know what the probability that we find no good photodiodes in n pairs of diodes. Testing each pair of diodes is an independent trial such that with probability p, both diodes of a pair are bad. From Problem 1.7.6, we can easily calculate p.

$$p = P\left[\text{both diodes are defective}\right] = P\left[D_1 D_2\right] = 6/25. \tag{1}$$

The probability of Z_n, the probability of zero acceptable diodes out of n pairs of diodes is p^n because on each test of a pair of diodes, both must be defective.

$$P\left[Z_n\right] = \prod_{i=1}^{n} p = p^n = \left(\frac{6}{25}\right)^n \tag{2}$$

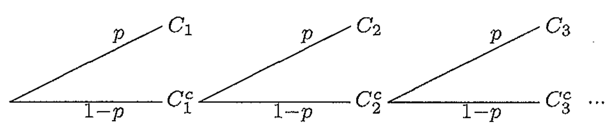
(b) Another way to phrase this question is to ask how many pairs must we test until $P[Z_n] \leq 0.01$. Since $P[Z_n] = (6/25)^n$, we require

$$\left(\frac{6}{25}\right)^n \leq 0.01 \quad \Rightarrow \quad n \geq \frac{\ln 0.01}{\ln 6/25} = 3.23. \tag{3}$$

Since n must be an integer, $n = 4$ pairs must be tested.

Problem 1.7.10 Solution

The experiment ends as soon as a fish is caught. The tree resembles



From the tree, $P[C_1] = p$ and $P[C_2] = (1-p)p$. Finally, a fish is caught on the nth cast if no fish were caught on the previous $n-1$ casts. Thus,

$$P\left[C_n\right] = (1-p)^{n-1} p. \tag{1}$$

Problem 1.8.1 Solution

There are $2^5 = 32$ different binary codes with 5 bits. The number of codes with exactly 3 zeros equals the number of ways of choosing the bits in which those zeros occur. Therefore there are $\binom{5}{3} = 10$ codes with exactly 3 zeros.

Problem 1.8.2 Solution

Since each letter can take on any one of the 4 possible letters in the alphabet, the number of 3 letter words that can be formed is $4^3 = 64$. If we allow each letter to appear only once then we have 4 choices for the first letter and 3 choices for the second and two choices for the third letter. Therefore, there are a total of $4 \cdot 3 \cdot 2 = 24$ possible codes.