ECE 5510 Fall 2009: Homework 3 Solutions

1. Y&G 2.5.9: see attached pages.
2. Y&G 2.6.2: see attached pages.
3. Y&G 3.2.4: see attached pages.
4. (a) Answer: $P[0 \le V \le 10] = F_V(10) - F_V(0) = 1 - \frac{(5)^2}{144} = 119/144 \approx 0.83$.

   (b) To find $E_V[V]$, first, find $f_V(v)$. For $-5 \le v < 7$,

   $$f_V(v) = \frac{v+5}{72}, \quad \text{for } -5 \le v < 7 \text{ and } 0 \text{ o.w.}$$

   Then,

   $$E_V[V] = \int_{v=-5}^{7} \frac{v^2 + 5v}{72} dv = \left. \frac{2v^3 + 15v^2}{432} \right|_{v=-5}^{7} = \frac{686 + 735 - (-250 + 375)}{432} = 1296/432 = 3$$

   Since it was not stated on the original handout, you could equally well have assumed that V was discrete. In this case, the answer for part (a) would be the same; while for part (b), we would have $P_V(v) = F_V(v) - F_V(v-1) = \frac{2v+9}{144}$ for $v = -4, \ldots, 7$, which result in

   $$E_V[V] = \sum_{v=-4}^{7} \frac{2v^2 + 9v}{144} = \frac{2(170) + 9(18)}{144} \approx 3.49.$$

   Either result is acceptable.

5. To use the $\Phi(\cdot)$ function, we need a standard normal r.v., which we obtain by subtracting the mean and dividing by the standard deviation:

   $$P[0 \le X \le 9] = P[0 - 5 \le X - 5 \le 9 - 5] = P\left[\frac{-5}{3} \le \frac{X-5}{3} \le \frac{4}{3}\right]$$

   Now, we can simplify using the CDF of the standard normal CDF:

   $$P\left[\frac{-5}{3} \le \frac{X-5}{3} \le \frac{4}{3}\right] = P\left[\frac{X-5}{3} \le \frac{4}{3}\right] - P\left[\frac{X-5}{3} \le \frac{-5}{3}\right] = \Phi(4/3) - \Phi(-5/3)$$

   Matlab returns 0.8610.

Problem 2.5.8 Solution

The following experiments are based on a common model of packet transmissions in data networks. In these networks, each data packet contains a cylic redundancy check (CRC) code that permits the receiver to determine whether the packet was decoded correctly. In the following, we assume that a packet is corrupted with probability $\epsilon = 0.001$, independent of whether any other packet is corrupted.

(a) Let $X = 1$ if a data packet is decoded correctly; otherwise $X = 0$. Random variable X is a Bernoulli random variable with PMF

$$P_X(x) = \begin{cases} 0.001 & x = 0 \\ 0.999 & x = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The parameter $\epsilon = 0.001$ is the probability a packet is corrupted. The expected value of X is

$$E[X] = 1 - \epsilon = 0.999 \tag{2}$$

(b) Let Y denote the number of packets received in error out of 100 packets transmitted. Y has the binomial PMF

$$P_Y(y) = \begin{cases} \binom{100}{y}(0.001)^y(0.999)^{100-y} & y = 0, 1, \ldots, 100 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The expected value of Y is

$$E[Y] = 100\epsilon = 0.1 \tag{4}$$

(c) Let L equal the number of packets that must be received to decode 5 packets in error. L has the Pascal PMF

$$P_L(l) = \begin{cases} \binom{l-1}{4}(0.001)^5(0.999)^{l-5} & l = 5, 6, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

The expected value of L is

$$E[L] = \frac{5}{\epsilon} = \frac{5}{0.001} = 5000 \tag{6}$$

(d) If packet arrivals obey a Poisson model with an average arrival rate of 1000 packets per second, then the number N of packets that arrive in 5 seconds has the Poisson PMF

$$P_N(n) = \begin{cases} 5000^n e^{-5000}/n! & n = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

The expected value of N is $E[N] = 5000$.

Problem 2.5.9 Solution

In this "double-or-nothing" type game, there are only two possible payoffs. The first is zero dollars, which happens when we lose 6 straight bets, and the second payoff is 64 dollars which happens unless we lose 6 straight bets. So the PMF of Y is

$$P_Y(y) = \begin{cases} (1/2)^6 = 1/64 & y = 0 \\ 1 - (1/2)^6 = 63/64 & y = 64 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The expected amount you take home is

$$E[Y] = 0(1/64) + 64(63/64) = 63 \tag{2}$$

So, on the average, we can expect to break even, which is not a very exciting proposition.

(b) From the PMF, it is straighforward to write down the CDF.

$$F_U(u) = \begin{cases} 0 & u < 1 \\ 1/4 & 1 \le u < 4 \\ 1/2 & 4 \le u < 9 \\ 1 & u \ge 9 \end{cases} \tag{5}$$

(c) From Definition 2.14, the expected value of U is

$$E[U] = \sum_u u P_U(u) = 1(1/4) + 4(1/4) + 9(1/2) = 5.75 \tag{6}$$

From Theorem 2.10, we can calculate the expected value of U as

$$E[U] = E\left[Y^2\right] = \sum_y y^2 P_Y(y) = 1^2(1/4) + 2^2(1/4) + 3^2(1/2) = 5.75 \tag{7}$$

As we expect, both methods yield the same answer.

Problem 2.6.2 Solution

From the solution to Problem 2.4.2, the PMF of X is

$$P_X(x) = \begin{cases} 0.2 & x = -1 \\ 0.5 & x = 0 \\ 0.3 & x = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The PMF of $V = |X|$ satisfies

$$P_V(v) = P[|X| = v] = P_X(v) + P_X(-v) \tag{2}$$

In particular,

$$P_V(0) = P_X(0) = 0.5 \qquad P_V(1) = P_X(-1) + P_X(1) = 0.5 \tag{3}$$

The complete expression for the PMF of V is

$$P_V(v) = \begin{cases} 0.5 & v = 0 \\ 0.5 & v = 1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(b) From the PMF, we can construct the staircase CDF of V.

$$F_V(v) = \begin{cases} 0 & v < 0 \\ 0.5 & 0 \le v < 1 \\ 1 & v \ge 1 \end{cases} \tag{5}$$

(c) From the PMF $P_V(v)$, the expected value of V is

$$E[V] = \sum_v P_V(v) = 0(1/2) + 1(1/2) = 1/2 \tag{6}$$

You can also compute $E[V]$ directly by using Theorem 2.10.

Problem 3.2.4 Solution

For $x < 0$, $F_X(x) = 0$. For $x \geq 0$,

$$F_X(x) = \int_0^x f_X(y)\, dy \tag{1}$$

$$= \int_0^x a^2 y e^{-a^2 y^2/2}\, dy \tag{2}$$

$$= -e^{-a^2 y^2/2}\Big|_0^x = 1 - e^{-a^2 x^2/2} \tag{3}$$

A complete expression for the CDF of X is

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-a^2 x^2/2} & x \geq 0 \end{cases} \tag{4}$$

Problem 3.2.5 Solution

$$f_X(x) = \begin{cases} ax^2 + bx & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

First, we note that a and b must be chosen such that the above PDF integrates to 1.

$$\int_0^1 (ax^2 + bx)\, dx = a/3 + b/2 = 1 \tag{2}$$

Hence, $b = 2 - 2a/3$ and our PDF becomes

$$f_X(x) = x(ax + 2 - 2a/3) \tag{3}$$

For the PDF to be non-negative for $x \in [0, 1]$, we must have $ax + 2 - 2a/3 \geq 0$ for all $x \in [0, 1]$. This requirement can be written as

$$a(2/3 - x) \leq 2 \qquad (0 \leq x \leq 1) \tag{4}$$

For $x = 2/3$, the requirement holds for all a. However, the problem is tricky because we must consider the cases $0 \leq x < 2/3$ and $2/3 < x \leq 1$ separately because of the sign change of the inequality. When $0 \leq x < 2/3$, we have $2/3 - x > 0$ and the requirement is most stringent at $x = 0$ where we require $2a/3 \leq 2$ or $a \leq 3$. When $2/3 < x \leq 1$, we can write the constraint as $a(x - 2/3) \geq -2$. In this case, the constraint is most stringent at $x = 1$, where we must have $a/3 \geq -2$ or $a \geq -6$. Thus a complete expression for our requirements are

$$-6 \leq a \leq 3 \qquad b = 2 - 2a/3 \tag{5}$$

As we see in the following plot, the shape of the PDF $f_X(x)$ varies greatly with the value of a.