ECE 5510 Fall 2009: Homework 5 Solutions

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6. **Probability of Real Roots of a Quadratic Equation**: You pick two real numbers A_1 and A_2 independently, both chosen to be uniform continuous random variables on $[-n, n]$.

   (a) Figure 1 shows the area of (A_1, A_2) for which the roots of the equation are real.

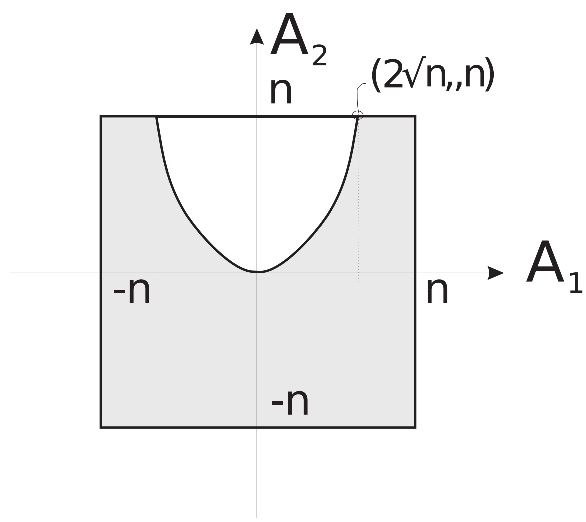


Figure 1: Area for 5(a).

   (b) For A_1, A_2 chosen independently, both as uniform r.v.s on $[-n, n]$, the joint pdf is just the product of the two uniform pdfs on $[-n, n]$,

$$f_{A_1,A_2}(a_1, a_2) = \begin{cases} \frac{1}{4n^2}, & -n \le a_1 \le n, \text{ and } -n \le a_2 \le n \\ 0, & o.w. \end{cases}$$

   For the case of $n > 4$ it is easiest to calculate one minus the area above the parabola (see

Figure from part (a)):

$$\begin{aligned}
P\left[t \text{ has real roots}\right] &= P\left[A_1^2 > 4A_2\right] \\
&= 1 - \int_{a_1=-2\sqrt{n}}^{2\sqrt{n}} \int_{a_2=a_1^2/4}^{n} \frac{1}{4n^2} da_2 da_1 \\
&= 1 - \frac{1}{4n^2} \int_{a_1=-2\sqrt{n}}^{2\sqrt{n}} \left[n - \frac{a_1^2}{4}\right] da_1 \\
&= 1 - \frac{1}{4n^2} \left[n a_1 - \frac{a_1^3}{12}\right|_{a_1=-2\sqrt{n}}^{2\sqrt{n}} \\
&= 1 - \frac{1}{4n^2} \left[\left(\frac{4n^{3/2}}{3}\right) - \left(-\frac{4n^{3/2}}{3}\right)\right] \\
&= 1 - \frac{2}{3\sqrt{n}} \qquad (1)
\end{aligned}$$

(c) We can see that as $n \to \infty$, $\sqrt{n}$ also approaches ∞. Thus the fraction in (1) goes to zero, thus the probability that t has real roots approaches 1.

Problem Solutions – Chapter 4

Problem 4.1.1 Solution

(a) The probability $P[X \leq 2, Y \leq 3]$ can be found be evaluating the joint CDF $F_{X,Y}(x,y)$ at $x = 2$ and $y = 3$. This yields

$$P[X \leq 2, Y \leq 3] = F_{X,Y}(2,3) = (1 - e^{-2})(1 - e^{-3}) \quad (1)$$

(b) To find the marginal CDF of X, $F_X(x)$, we simply evaluate the joint CDF at $y = \infty$.

$$F_X(x) = F_{X,Y}(x, \infty) = \begin{cases} 1 - e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

(c) Likewise for the marginal CDF of Y, we evaluate the joint CDF at $X = \infty$.

$$F_Y(y) = F_{X,Y}(\infty, y) = \begin{cases} 1 - e^{-y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

Problem 4.1.2 Solution

(a) Because the probability that any random variable is less than $-\infty$ is zero, we have

$$F_{X,Y}(x, -\infty) = P[X \leq x, Y \leq -\infty] \leq P[Y \leq -\infty] = 0 \quad (1)$$

(b) The probability that any random variable is less than infinity is always one.

$$F_{X,Y}(x, \infty) = P[X \leq x, Y \leq \infty] = P[X \leq x] = F_X(x) \quad (2)$$

(c) Although $P[Y \leq \infty] = 1$, $P[X \leq -\infty] = 0$. Therefore the following is true.

$$F_{X,Y}(-\infty, \infty) = P[X \leq -\infty, Y \leq \infty] \leq P[X \leq -\infty] = 0 \quad (3)$$

(d) Part (d) follows the same logic as that of part (a).

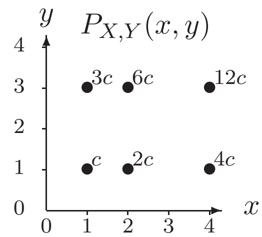
$$F_{X,Y}(-\infty, y) = P[X \leq -\infty, Y \leq y] \leq P[X \leq -\infty] = 0 \quad (4)$$

(e) Analogous to Part (b), we find that

$$F_{X,Y}(\infty, y) = P[X \leq \infty, Y \leq y] = P[Y \leq y] = F_Y(y) \quad (5)$$

Problem 4.2.1 Solution

In this problem, it is helpful to label points with nonzero probability on the X, Y plane:



(a) We must choose c so the PMF sums to one:

$$\sum_{x=1,2,4} \sum_{y=1,3} P_{X,Y}(x,y) = c \sum_{x=1,2,4} x \sum_{y=1,3} y \tag{1}$$

$$= c\,[1(1+3) + 2(1+3) + 4(1+3)] = 28c \tag{2}$$

Thus $c = 1/28$.

(b) The event $\{Y < X\}$ has probability

$$P[Y < X] = \sum_{x=1,2,4} \sum_{y<x} P_{X,Y}(x,y) = \frac{1(0) + 2(1) + 4(1+3)}{28} = \frac{18}{28} \tag{3}$$

(c) The event $\{Y > X\}$ has probability

$$P[Y > X] = \sum_{x=1,2,4} \sum_{y>x} P_{X,Y}(x,y) = \frac{1(3) + 2(3) + 4(0)}{28} = \frac{9}{28} \tag{4}$$

(d) There are two ways to solve this part. The direct way is to calculate

$$P[Y = X] = \sum_{x=1,2,4} \sum_{y=x} P_{X,Y}(x,y) = \frac{1(1) + 2(0)}{28} = \frac{1}{28} \tag{5}$$

The indirect way is to use the previous results and the observation that

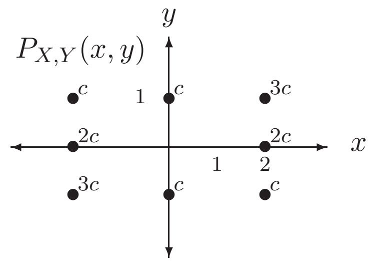
$$P[Y = X] = 1 - P[Y < X] - P[Y > X] = (1 - 18/28 - 9/28) = 1/28 \tag{6}$$

(e)

$$P[Y = 3] = \sum_{x=1,2,4} P_{X,Y}(x,3) = \frac{(1)(3) + (2)(3) + (4)(3)}{28} = \frac{21}{28} = \frac{3}{4} \tag{7}$$

Problem 4.2.2 Solution

On the X, Y plane, the joint PMF is



(a) To find c, we sum the PMF over all possible values of X and Y. We choose c so the sum equals one.

$$\sum_x \sum_y P_{X,Y}(x,y) = \sum_{x=-2,0,2} \sum_{y=-1,0,1} c\,|x+y| = 6c + 2c + 6c = 14c \tag{1}$$

Thus $c = 1/14$.

(b)

$$P[Y < X] = P_{X,Y}(0,-1) + P_{X,Y}(2,-1) + P_{X,Y}(2,0) + P_{X,Y}(2,1) \tag{2}$$
$$= c + c + 2c + 3c = 7c = 1/2 \tag{3}$$

(c)

$$P[Y > X] = P_{X,Y}(-2,-1) + P_{X,Y}(-2,0) + P_{X,Y}(-2,1) + P_{X,Y}(0,1) \tag{4}$$
$$= 3c + 2c + c + c = 7c = 1/2 \tag{5}$$

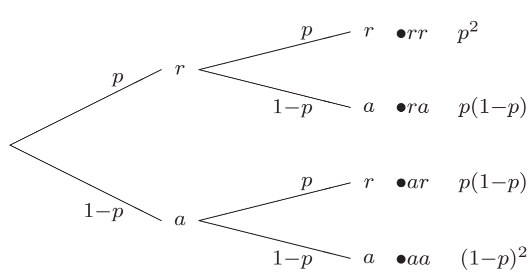
(d) From the sketch of $P_{X,Y}(x,y)$ given above, $P[X = Y] = 0$.

(e)

$$P[X < 1] = P_{X,Y}(-2,-1) + P_{X,Y}(-2,0) + P_{X,Y}(-2,1)$$
$$+ P_{X,Y}(0,-1) + P_{X,Y}(0,1) \tag{6}$$
$$= 8c = 8/14. \tag{7}$$

Problem 4.2.3 Solution

Let r (reject) and a (accept) denote the result of each test. There are four possible outcomes: rr, ra, ar, aa. The sample tree is



Now we construct a table that maps the sample outcomes to values of X and Y.

outcome	$P[\cdot]$	X	Y
rr	p^2	1	1
ra	$p(1-p)$	1	0
ar	$p(1-p)$	0	1
aa	$(1-p)^2$	0	0

(1)

This table is esentially the joint PMF $P_{X,Y}(x, y)$.

$$P_{X,Y}(x,y) = \begin{cases} p^2 & x=1, y=1 \\ p(1-p) & x=0, y=1 \\ p(1-p) & x=1, y=0 \\ (1-p)^2 & x=0, y=0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Problem 4.2.4 Solution

The sample space is the set $S = \{hh, ht, th, tt\}$ and each sample point has probability 1/4. Each sample outcome specifies the values of X and Y as given in the following table

outcome	X	Y
hh	0	1
ht	1	0
th	1	1
tt	2	0

(1)

The joint PMF can represented by the table

$P_{X,Y}(x,y)$	$y=0$	$y=1$
$x=0$	0	1/4
$x=1$	1/4	1/4
$x=2$	1/4	0

(2)

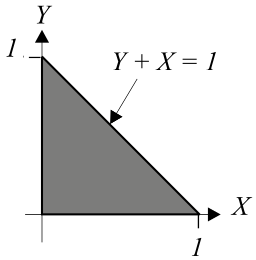
Problem 4.2.5 Solution

As the problem statement says, reasonable arguments can be made for the labels being X and Y or x and y. As we see in the arguments below, the lowercase choice of the text is somewhat arbitrary.

- *Lowercase axis labels:* For the lowercase labels, we observe that we are depicting the masses associated with the joint PMF $P_{X,Y}(x, y)$ whose arguments are x and y. Since the PMF function is defined in terms of x and y, the axis labels should be x and y.

- *Uppercase axis labels:* On the other hand, we are depicting the possible outcomes (labeled with their respective probabilities) of the pair of random variables X and Y. The corresponding axis labels should be X and Y just as in Figure 4.2. The fact that we have labeled the possible outcomes by their probabilities is irrelevant. Further, since the expression for the PMF $P_{X,Y}(x, y)$ given in the figure could just as well have been written $P_{X,Y}(\cdot, \cdot)$, it is clear that the lowercase x and y are not what matter.

(a) The joint PDF of X and Y is



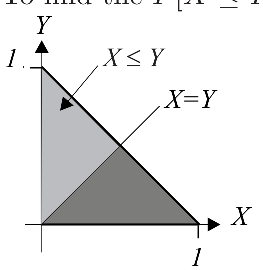
$$f_{X,Y}(x,y) = \begin{cases} c & x+y \leq 1, x, y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

To find the constant c we integrate over the region shown. This gives

$$\int_0^1 \int_0^{1-x} c\, dy\, dx = cx - \frac{cx}{2}\Big|_0^1 = \frac{c}{2} = 1 \tag{2}$$

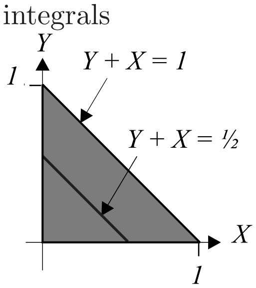
Therefore $c = 2$.

(b) To find the $P[X \leq Y]$ we look to integrate over the area indicated by the graph



$$P[X \leq Y] = \int_0^{1/2} \int_x^{1-x} dy\, dx \tag{3}$$
$$= \int_0^{1/2} (2-4x)\, dx \tag{4}$$
$$= 1/2 \tag{5}$$

(c) The probability $P[X + Y \leq 1/2]$ can be seen in the figure. Here we can set up the following integrals



$$P[X + Y \leq 1/2] = \int_0^{1/2} \int_0^{1/2-x} 2\, dy\, dx \tag{6}$$
$$= \int_0^{1/2} (1-2x)\, dx \tag{7}$$
$$= 1/2 - 1/4 = 1/4 \tag{8}$$

Problem 4.4.2 Solution

Given the joint PDF

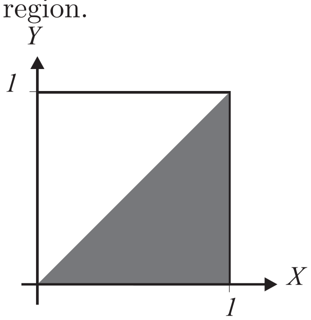
$$f_{X,Y}(x,y) = \begin{cases} cxy^2 & 0 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) To find the constant c integrate $f_{X,Y}(x,y)$ over the all possible values of X and Y to get

$$1 = \int_0^1 \int_0^1 cxy^2\, dx\, dy = c/6 \tag{2}$$

Therefore $c = 6$.

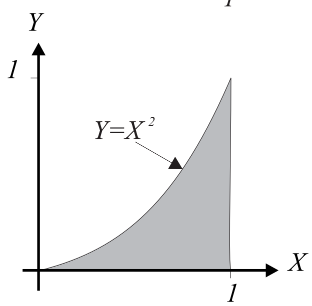
(b) The probability $P[X \geq Y]$ is the integral of the joint PDF $f_{X,Y}(x,y)$ over the indicated shaded region.



$$P[X \geq Y] = \int_0^1 \int_0^x 6xy^2 \, dy \, dx \tag{3}$$

$$= \int_0^1 2x^4 \, dx \tag{4}$$

$$= 2/5 \tag{5}$$

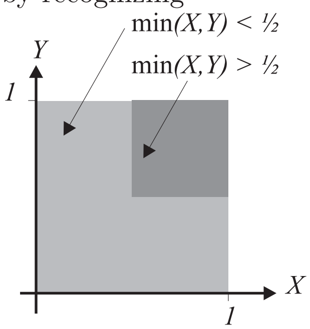


Similarly, to find $P[Y \leq X^2]$ we can integrate over the region shown in the figure.

$$P\left[Y \leq X^2\right] = \int_0^1 \int_0^{x^2} 6xy^2 \, dy \, dx \tag{6}$$

$$= 1/4 \tag{7}$$

(c) Here we can choose to either integrate $f_{X,Y}(x,y)$ over the lighter shaded region, which would require the evaluation of two integrals, or we can perform one integral over the darker region by recognizing

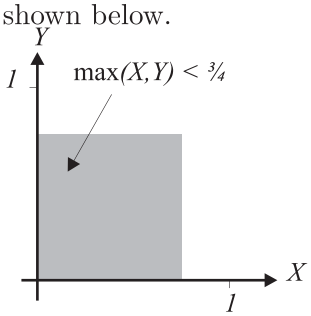


$$P[\min(X,Y) \leq 1/2] = 1 - P[\min(X,Y) > 1/2] \tag{8}$$

$$= 1 - \int_{1/2}^1 \int_{1/2}^1 6xy^2 \, dx \, dy \tag{9}$$

$$= 1 - \int_{1/2}^1 \frac{9y^2}{4} \, dy = \frac{11}{32} \tag{10}$$

(d) The probability $P[\max(X,Y) \leq 3/4]$ can be found be integrating over the shaded region shown below.



$$P[\max(X,Y) \leq 3/4] = P[X \leq 3/4, Y \leq 3/4] \tag{11}$$

$$= \int_0^{\frac{3}{4}} \int_0^{\frac{3}{4}} 6xy^2 \, dx \, dy \tag{12}$$

$$= \left(x^2 \Big|_0^{3/4} \right) \left(y^3 \Big|_0^{3/4} \right) \tag{13}$$

$$= (3/4)^5 = 0.237 \tag{14}$$

Problem 4.4.3 Solution

The joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} 6e^{-(2x+3y)} & x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) Random variable $W = 2^{XY}$ has expected value

$$E\left[2^{XY}\right] = \sum_{x=-2,0,2} \sum_{y=-1,0,1} 2^{xy} P_{X,Y}(x,y) \quad (3)$$

$$= 2^{-2(-1)}\frac{3}{14} + 2^{-2(0)}\frac{2}{14} + 2^{-2(1)}\frac{1}{14} + 2^{0(-1)}\frac{1}{14} + 2^{0(1)}\frac{1}{14} \quad (4)$$

$$+ 2^{2(-1)}\frac{1}{14} + 2^{2(0)}\frac{2}{14} + 2^{2(1)}\frac{3}{14} \quad (5)$$

$$= 61/28 \quad (6)$$

(b) The correlation of X and Y is

$$r_{X,Y} = \sum_{x=-2,0,2} \sum_{y=-1,0,1} xyP_{X,Y}(x,y) \quad (7)$$

$$= \frac{-2(-1)(3)}{14} + \frac{-2(0)(2)}{14} + \frac{-2(1)(1)}{14} + \frac{2(-1)(1)}{14} + \frac{2(0)(2)}{14} + \frac{2(1)(3)}{14} \quad (8)$$

$$= 4/7 \quad (9)$$

(c) The covariance of X and Y is

$$\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = 4/7 \quad (10)$$

(d) The correlation coefficient is

$$\rho_{X,Y} = \frac{\text{Cov}[X,Y]}{\sqrt{\text{Var}[X]\,\text{Var}[Y]}} = \frac{2}{\sqrt{30}} \quad (11)$$

(e) By Theorem 4.16,

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X,Y] \quad (12)$$

$$= \frac{24}{7} + \frac{5}{7} + 2\frac{4}{7} = \frac{37}{7}. \quad (13)$$

Problem 4.7.3 Solution

In the solution to Quiz 4.3, the joint PMF and the marginal PMFs are

$P_{H,B}(h,b)$	$b=0$	$b=2$	$b=4$	$P_H(h)$
$h=-1$	0	0.4	0.2	0.6
$h=0$	0.1	0	0.1	0.2
$h=1$	0.1	0.1	0	0.2
$P_B(b)$	0.2	0.5	0.3	

(1)

From the joint PMF, the correlation coefficient is

$$r_{H,B} = E[HB] = \sum_{h=-1}^{1} \sum_{b=0,2,4} hbP_{H,B}(h,b) \quad (2)$$

$$= -1(2)(0.4) + 1(2)(0.1) + -1(4)(0.2) + 1(4)(0) \quad (3)$$

$$= -1.4 \quad (4)$$

since only terms in which both h and b are nonzero make a contribution. Using the marginal PMFs, the expected values of X and Y are

$$E[H] = \sum_{h=-1}^{1} hP_H(h) = -1(0.6) + 0(0.2) + 1(0.2) = -0.2 \quad (5)$$
$$E[B] = \sum_{b=0,2,4} bP_B(b) = 0(0.2) + 2(0.5) + 4(0.3) = 2.2 \quad (6)$$

The covariance is

$$\text{Cov}[H,B] = E[HB] - E[H]E[B] = -1.4 - (-0.2)(2.2) = -0.96 \quad (7)$$

Problem 4.7.4 Solution

From the joint PMF, $P_X(x)Y$, found in Example 4.13, we can find the marginal PMF for X or Y by summing over the columns or rows of the joint PMF.

$$P_Y(y) = \begin{cases} 25/48 & y=1 \\ 13/48 & y=2 \\ 7/48 & y=3 \\ 3/48 & y=4 \\ 0 & \text{otherwise} \end{cases} \qquad P_X(x) = \begin{cases} 1/4 & x=1,2,3,4 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

(a) The expected values are

$$E[Y] = \sum_{y=1}^{4} yP_Y(y) = 1\frac{25}{48} + 2\frac{13}{48} + 3\frac{7}{48} + 4\frac{3}{48} = 7/4 \quad (2)$$
$$E[X] = \sum_{x=1}^{4} xP_X(x) = \frac{1}{4}(1+2+3+4) = 5/2 \quad (3)$$

(b) To find the variances, we first find the second moments.

$$E\left[Y^2\right] = \sum_{y=1}^{4} y^2 P_Y(y) = 1^2\frac{25}{48} + 2^2\frac{13}{48} + 3^2\frac{7}{48} + 4^2\frac{3}{48} = 47/12 \quad (4)$$
$$E\left[X^2\right] = \sum_{x=1}^{4} x^2 P_X(x) = \frac{1}{4}\left(1^2+2^2+3^2+4^2\right) = 15/2 \quad (5)$$

Now the variances are

$$\text{Var}[Y] = E\left[Y^2\right] - (E[Y])^2 = 47/12 - (7/4)^2 = 41/48 \quad (6)$$
$$\text{Var}[X] = E\left[X^2\right] - (E[X])^2 = 15/2 - (5/2)^2 = 5/4 \quad (7)$$