ECE 5510 Fall 2009: Homework 6 Solutions

1. Y&G 4.7.7: See additional pages.
2. Y&G 4.11.2: See additional pages.
3. (Double credit) The shape of the pdf is drawn in Fig. 1, for your reference (it was not asked for in the problem, but it may help.)

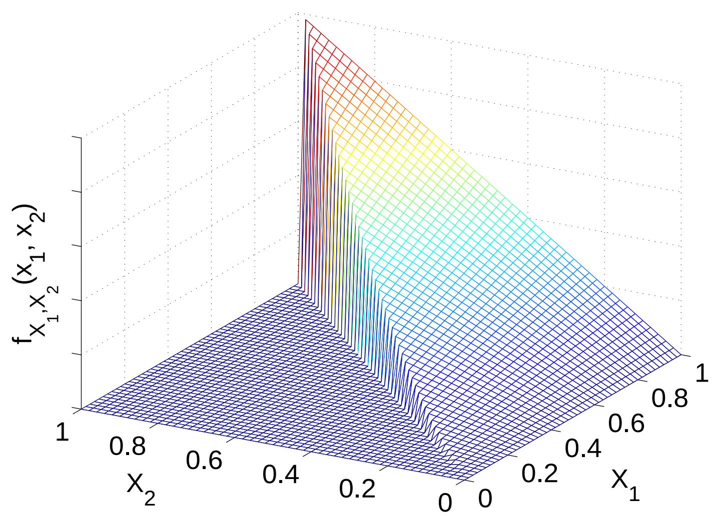


Figure 1: Figure for problem 3 of the joint pdf of X_1 and X_2.

(a) Since the volume under the joint pdf is one,

$$\begin{aligned} 1 &= \int\int f_{X_1,X_2}(x_1, x_2)\, dx_1 dx_2 = \int_{x_1=0}^{1} \int_{x_2=0}^{x_1} c x_1 x_2 \, dx_2 dx_1 \\ &= c \int_{x_1=0}^{1} x_1 \int_{x_2=0}^{x_1} x_2 \, dx_2 dx_1 = c \int_{x_1=0}^{1} x_1 (x_1^2/2)\, dx_1 \\ &= \frac{c}{2} \int_{x_1=0}^{1} x_1^3 \, dx_1 = \frac{c}{2} \left[x_1^4/4 \right|_{x_1=0}^{1} = \frac{c}{8}, \end{aligned}$$

which implies that $c = 8$.

(b) Random variables X_1 and X_2 are <u>not</u> independent. One simple explanation is that if the support (the non-zero part) of the pdf is non-square, then the two r.v.s are not independent. Alternatively, you could show that the product of the marginal pdfs is not equal to the joint pdf.

(c) $f_{X_1}(x_1) = \int_{x_2=0}^{x_1} 8 x_1 x_2 \, dx_2 = 8 x_1 (x_1^2/2) = 4 x_1^3$, for $0 < x_1 < 1$, and zero otherwise.

(d) Find $f_{X_2|X_1}(x_2|0.5)$. First, find $f_{X_2|X_1}(x_2|x_1)$:

$$\begin{aligned} f_{X_2|X_1}(x_2|x_1) &= \begin{cases} \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_1}(x_1)}, & 0 < x_2 < x_1 < 1 \\ 0, & o.w. \end{cases} \\ &= \begin{cases} \frac{8x_1x_2}{4x_1^3}, & 0 < x_2 < x_1 < 1 \\ 0, & o.w. \end{cases} = \begin{cases} \frac{2x_2}{x_1^2}, & 0 < x_2 < x_1 < 1 \\ 0, & o.w. \end{cases} \end{aligned}$$

so at $x_1 = 0.5$, the conditional pdf is

$$f_{X_2|X_1}(x_2|0.5) = \begin{cases} 8x_2, & 0 < x_2 < 0.5 \\ 0, & o.w. \end{cases}$$

(e) To compute the variance of X_1, first find its mean:

$$E_{X_1}[X_1] = \int x_1 f_{X_1}(x_1) dx_1 = \int_{x_1=0}^{1} x_1 4x_1^3 dx_1 = 4/5$$

Next, find the 2nd moment of X_1,

$$E_{X_1}\left[X_1^2\right] = \int x_1^2 f_{X_1}(x_1) dx_1 = \int_{x_1=0}^{1} x_1^2 4x_1^3 dx_1 = 2/3$$

Finally, the variance is

$$\text{Var}_{X_1}[X_1] = E_{X_1}\left[X_1^2\right] - (E_{X_1}[X_1])^2 = 2/3 - 16/25 = \frac{2}{75} \approx 0.0267$$

(f) This expectation of the product of X_1X_2 is

$$\begin{aligned} E_{X_1,X_2}[X_1X_2] = &= \int\int x_1x_2 f_{X_1,X_2}(x_1,x_2)\, dx_1dx_2 = \int_{x_1=0}^{1}\int_{x_2=0}^{x_1} 8x_1^2x_2^2\, dx_2dx_1 \\ &= 8\int_{x_1=0}^{1} x_1^2 \int_{x_2=0}^{x_1} x_2^2\, dx_2dx_1 = 8\int_{x_1=0}^{1} x_1^2(x_1^3/3)\, dx_1 \\ &= \frac{8}{3}\int_{x_1=0}^{1} x_1^5\, dx_1 = \frac{8}{3}\left[x_1^6/6\right|_{x_1=0}^{1} = \frac{8}{18} = \frac{4}{9}. \end{aligned}$$

(g) For the covariance matrix, we need two additional numbers, the variance of X_2 and $\text{Cov}(X_1, X_2)$. For the latter,

$$\text{Cov}(X_1,X_2) = E_{X_1,X_2}[X_1X_2] - E_{X_1}[X_1]\, E_{X_2}[X_2]$$

Finding the marginal pdf of X_2, $f_{X_2}(x_2) = \int_{x_1=x_2}^{1} 8x_1x_2\, dx_1 = 8x_2\left(\frac{1}{2} - \frac{x_2^2}{2}\right) = 4(x_2 - x_2^3)$, for $0 < x_2 < 1$, and zero otherwise. So

$$\begin{aligned} E_{X_2}[X_2] &= \int_{x_2=0}^{1} x_2 4(x_2 - x_2^3)dx_2 = 4\left[\frac{x_2^3}{3} - \frac{x_2^5}{5}\right|_0^1 = 4\left[\frac{1}{3} - \frac{1}{5}\right] = 8/15 \\ E_{X_2}\left[X_2^2\right] &= \int_{x_2=0}^{1} x_2^2 4(x_2 - x_2^3)dx_2 = 4\left[\frac{x_2^4}{4} - \frac{x_2^6}{6}\right|_0^1 = 4\left[\frac{1}{4} - \frac{1}{6}\right] = 1/3 \\ \text{Var}_{X_2}[X_2] &= E_{X_2}\left[X_2^2\right] - (E_{X_2}[X_2])^2 = 1/3 - (8/15)^2 = 11/225 \approx 0.0489 \\ \text{Cov}(X_1,X_2) &= E_{X_1,X_2}[X_1X_2] - E_{X_1}[X_1]\, E_{X_2}[X_2] \\ &= \frac{4}{9} - \frac{4}{5}\frac{8}{15} = \frac{4}{225} \approx 0.0178 \qquad (1) \end{aligned}$$

So the covariance matrix can be formed with the covariance and variance terms:

$$C_{\mathbf{X}} = \begin{bmatrix} \frac{6}{225} & \frac{4}{225} \\ \frac{4}{225} & \frac{11}{225} \end{bmatrix} \approx \begin{bmatrix} 0.0267 & 0.0178 \\ 0.0178 & 0.0489 \end{bmatrix}$$

A good check to do is to make sure that the correlation coefficient has magnitude less than one. (This was not asked for but is a good thing to do.)

$$\rho = \frac{\mathrm{Cov}\,(X_1, X_2)}{\sqrt{\mathrm{Var}_{X_2}[X_2]\,\mathrm{Var}_{X_1}[X_1]}} = \frac{4}{\sqrt{66}} \approx 0.492$$

So yes, the ρ is valid.

4. (a) In Y&G page 192-193, it points out to us that we can write the bivariate Gaussian pdf as:

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}} \frac{1}{\sqrt{2\pi\tilde{\sigma}_2^2}} e^{-\frac{(x_2-\tilde{\mu}_2(x_1))^2}{2\tilde{\sigma}_2^2}} \\ f_{X_2|X_1}(x_2|x_1) &= f_{X_1,X_2}(x_1,x_2)/f_{X_1}(x_1) \\ &= \frac{1}{\sqrt{2\pi\tilde{\sigma}_2^2}} e^{-\frac{(x_2-\tilde{\mu}_2(x_1))^2}{2\tilde{\sigma}_2^2}} \end{aligned}$$

where

$$\begin{aligned} \tilde{\mu}_2(x_1) &= \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1 - \mu_1) \\ \tilde{\sigma}_2^2 &= \sigma_2^2(1-\rho^2) \end{aligned}$$

In this case, for the particular values, $\mu_{X_1} = 1$, $\mu_{X_2} = 2$, $\sigma_{X_1}^2 = 1$, $\sigma_{X_2}^2 = 4$ and $\rho = 0.5$:

$$\begin{aligned} \tilde{\mu}_2(x_1) &= 2 + 0.5\frac{2}{1}(x_1 - 1) = 1 + x_1 \\ \tilde{\sigma}_2^2 &= 4(1-0.25) = 3 \end{aligned}$$

So,

$$f_{X_2|X_1}(x_2|x_1) = \frac{1}{\sqrt{6\pi}} e^{-(x_2-x_1-1)^2/6}.$$

(b) Let $Z = 2X_1 - 3X_2$. First, find the mean and variance of Z:

$$\begin{aligned} E_{X_1,X_2}[Z] &= E_{X_1,X_2}[2X_1 - 3X_2] = 2E_{X_1,X_2}[X_1] - 3E_{X_1,X_2}[X_2] \\ &= 2(1) - 3(2) = -4 \\ \mathrm{Var}_{X_1,X_2}[Z] &= \mathrm{Var}_{X_1,X_2}[2X_1 - 3X_2] \\ &= \mathrm{Var}_{X_1}[2X_1] + \mathrm{Var}_{X_2}[-3X_2] + 2\mathrm{Cov}\,(2X_1, -3X_2) \qquad (2) \end{aligned}$$

But from the definition of covariance, we can see that

$$\begin{aligned} \mathrm{Cov}\,(2X_1, -3X_2) &= E_{X_1,X_2}[2X_1(-3)X_2] - E_{X_1}[2X_1]\,E_{X_2}[-3X_2] \\ &= -6E_{X_1,X_2}[X_1X_2] + 6E_{X_1}[X_1]\,E_{X_2}[X_2] \\ &= -6\mathrm{Cov}\,(X_1, X_2) \qquad (3) \end{aligned}$$

Also, from the definition of correlation coefficient ρ,

$$\mathrm{Cov}\left(X_1, X_2\right) = \rho\sqrt{\sigma_{X_1}^2 \sigma_{X_2}^2} = 0.5(2)(1) = 1.$$

Plugging this back into (2),

$$\mathrm{Var}_{X_1,X_2}[Z] = 4\mathrm{Var}_{X_1}[X_1] + 9\mathrm{Var}_{X_2}[X_2] + 2(-6)\mathrm{Cov}\left(X_1, X_2\right) = 4(1) + 9(4) - 12(1) = 28.$$

5. Since $P_{X_1}(x)$ is binomial with parameters n and p,

$$P_{X_1}(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

And the pmf for X_2 is exactly the same since X_2 is also Binomial with the same parameters. Thus given that $P_Y(y) = \sum_{x \in S_X} P_{X_1}(x) P_{X_2}(y-x)$,

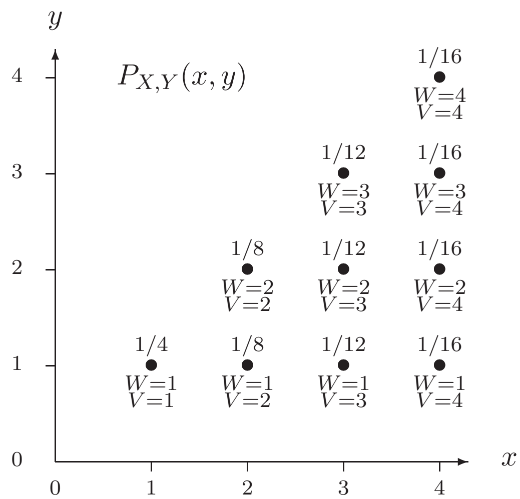
$$\begin{aligned} P_Y(y) &= \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} \binom{n}{y-x} p^{y-x} (1-p)^{n-y+x} \\ &= \sum_{x=0}^{n} \binom{n}{x} \binom{n}{y-x} p^x p^{y-x} (1-p)^{n-x} (1-p)^{n-y+x} \\ &= \sum_{x=0}^{n} \binom{n}{x} \binom{n}{y-x} p^y (1-p)^{2n-y} \\ &= p^y (1-p)^{2n-y} \sum_{x=0}^{n} \binom{n}{x} \binom{n}{y-x} \end{aligned}$$

Given that $\sum_{x=0}^{n} \binom{n}{x}\binom{n}{y-x} = \binom{2n}{y}$,

$$P_Y(y) = \binom{2n}{y} p^y (1-p)^{2n-y}$$

which is just *the binomial distribution for parameters p and $2n$*. This makes intuitive sense because if X_1 counts how many successes we have in n independent Bernoulli trials, and then we add that number to X_2, the number of successes we have in another n independent Bernoulli trials, it would be the same as counting the number of successes in $2n$ independent Bernoulli trials.

Problem 4.7.6 Solution



To solve this problem, we identify the values of $W = \min(X, Y)$ and $V = \max(X, Y)$ for each possible pair x, y. Here we observe that $W = Y$ and $V = X$. This is a result of the underlying experiment in that given $X = x$, each $Y \in \{1, 2, \ldots, x\}$ is equally likely. Hence $Y \leq X$. This implies $\min(X, Y) = Y$ and $\max(X, Y) = X$.

Using the results from Problem 4.7.4, we have the following answers.

(a) The expected values are

$$E[W] = E[Y] = 7/4 \qquad E[V] = E[X] = 5/2 \tag{1}$$

(b) The variances are

$$\mathrm{Var}[W] = \mathrm{Var}[Y] = 41/48 \qquad \mathrm{Var}[V] = \mathrm{Var}[X] = 5/4 \tag{2}$$

(c) The correlation is

$$r_{W,V} = E[WV] = E[XY] = r_{X,Y} = 5 \tag{3}$$

(d) The covariance of W and V is

$$\mathrm{Cov}[W, V] = \mathrm{Cov}[X, Y] = 10/16 \tag{4}$$

(e) The correlation coefficient is

$$\rho_{W,V} = \rho_{X,Y} = \frac{10/16}{\sqrt{(41/48)(5/4)}} \approx 0.605 \tag{5}$$

Problem 4.7.7 Solution

First, we observe that Y has mean $\mu_Y = a\mu_X + b$ and variance $\mathrm{Var}[Y] = a^2 \mathrm{Var}[X]$. The covariance of X and Y is

$$\begin{aligned} \mathrm{Cov}[X, Y] &= E[(X - \mu_X)(aX + b - a\mu_X - b)] && (1)\\ &= aE\left[(X - \mu_X)^2\right] && (2)\\ &= a\,\mathrm{Var}[X] && (3) \end{aligned}$$

The correlation coefficient is

$$\rho_{X,Y} = \frac{\mathrm{Cov}[X, Y]}{\sqrt{\mathrm{Var}[X]}\sqrt{\mathrm{Var}[Y]}} = \frac{a\,\mathrm{Var}[X]}{\sqrt{\mathrm{Var}[X]}\sqrt{a^2\,\mathrm{Var}[X]}} = \frac{a}{|a|} \tag{4}$$

When $a > 0$, $\rho_{X,Y} = 1$. When $a < 0$, $\rho_{X,Y} = 1$.

Problem 4.11.2 Solution

For the joint PDF

$$f_{X,Y}(x,y) = ce^{-(2x^2-4xy+4y^2)}, \tag{1}$$

we proceed as in Problem 4.11.1 to find values for σ_Y, σ_X, $E[X]$, $E[Y]$ and ρ.

(a) First, we try to solve the following equations

$$\left(\frac{x-E[X]}{\sigma_X}\right)^2 = 4(1-\rho^2)x^2 \tag{2}$$

$$\left(\frac{y-E[Y]}{\sigma_Y}\right)^2 = 8(1-\rho^2)y^2 \tag{3}$$

$$\frac{2\rho}{\sigma_X\sigma_Y} = 8(1-\rho^2) \tag{4}$$

The first two equations yield $E[X] = E[Y] = 0$

(b) To find the correlation coefficient ρ, we observe that

$$\sigma_X = 1/\sqrt{4(1-\rho^2)} \qquad \sigma_Y = 1/\sqrt{8(1-\rho^2)} \tag{5}$$

Using σ_X and σ_Y in the third equation yields $\rho = 1/\sqrt{2}$.

(c) Since $\rho = 1/\sqrt{2}$, now we can solve for σ_X and σ_Y.

$$\sigma_X = 1/\sqrt{2} \qquad \sigma_Y = 1/2 \tag{6}$$

(d) From here we can solve for c.

$$c = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} = \frac{2}{\pi} \tag{7}$$

(e) X and Y are dependent because $\rho \neq 0$.

Problem 4.11.3 Solution

From the problem statement, we learn that

$$\mu_X = \mu_Y = 0 \qquad \sigma_X^2 = \sigma_Y^2 = 1 \tag{1}$$

From Theorem 4.30, the conditional expectation of Y given X is

$$E[Y|X] = \tilde{\mu}_Y(X) = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(X-\mu_X) = \rho X \tag{2}$$

In the problem statement, we learn that $E[Y|X] = X/2$. Hence $\rho = 1/2$. From Definition 4.17, the joint PDF is

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{3\pi^2}}e^{-2(x^2-xy+y^2)/3} \tag{3}$$