ECE 5510 Fall 2009: Homework 7 Solutions

1. (a) Find the mean and standard deviation of $Y = 20 + \sum_{i=1}^{10} X_i$. For the mean, using the linearity of the expected value,

$$\begin{aligned} E_Y[Y] &= E_{\mathbf{X}}\left[20 + \sum_{i=1}^{10} X_i\right] \\ &= 20 + \sum_{i=1}^{10} E_{X_i}[X_i] = 20 + 10(2) = 40 \text{ minutes.} \end{aligned}$$

For the variance, because the X_i are independent, we know the covariance terms are zero, so that

$$\begin{aligned} \text{Var}_Y[Y] &= \text{Var}_{\mathbf{X}}\left[20 + \sum_{i=1}^{10} X_i\right] \\ &= \sum_{i=1}^{10} \text{Var}_{X_i}[X_i] = 10(0.5)^2 = 2.5 \text{ minutes}^2. \end{aligned}$$

The standard deviation is always the square root of the variance: $\sqrt{2.5} \approx 1.58$ minutes.

(b) Now, $Y = 30 + \sum_{i=1}^{5} X_i$, so

$$\begin{aligned} E_Y[Y] &= E_{\mathbf{X}}\left[30 + \sum_{i=1}^{5} X_i\right] \\ &= 30 + \sum_{i=1}^{5} E_{X_i}[X_i] = 30 + 5(2) = 40 \text{ minutes.} \\ \text{Var}_Y[Y] &= \text{Var}_{\mathbf{X}}\left[30 + \sum_{i=1}^{5} X_i\right] \\ &= \sum_{i=1}^{5} \text{Var}_{X_i}[X_i] = 5(0.5)^2 = 1.25 \text{ minutes}^2. \qquad (1) \end{aligned}$$

Thus the standard deviation is $\sqrt{1.25} \approx 1.12$ minutes.

(c) You and your boss have the same mean travel time, but your route has twice the standard deviation. Thus your travel time is more variable. Assuming that you both allow the same amount of time (for example, 42 minutes) to get to work, your boss will be more predictably 'on-time'. So the analysis does not back up your boss' claim.

2. Y&G 5.7.7: See attached pages.

3. Y&G 10.5.1: See attached pages.

4. Y&G 10.5.5: See attached pages.
5. Y&G 10.5.6: See attached pages.
6. Let $Y(t)$ be a Poisson process with arrival rate λ. Denote the first arrival time as T_1 and the second arrival time as T_2.

(a) Define $\Delta = T_2 - T_1$. Because $(0, T_1)$ and (T_1, T_2) are non-overlapping, by the independent increments property of Poisson processes, T_1 and Δ are independent, so that

$$f_{T_1,\Delta)}(t_1, \delta) = f_{T_1}(t_1) f_\Delta(\delta)$$

Both are Exponential with arrival rate λ, just with different durations of time, so, as long as $\delta \geq 0$,

$$f_{T_1,\Delta}(t_1, \delta) = \lambda e^{-\lambda t_1} \lambda e^{-\lambda \delta} = \lambda^2 e^{-\lambda(t_1+\delta)}$$

Since $T_2 = T_1 + \Delta$, for $\Delta \geq 0$, then if $T_2 = t_2$, we can use $\delta = t_2 - t_1$,

$$f_{T_1,T_2}(t_1, t_2) = f_{T_1,\Delta}(t_1, t_2 - t_1) == \lambda^2 e^{-\lambda(t_1+t_2-t_1)} = \lambda^2 e^{-\lambda t_2}$$

Since $\delta \geq 0$, we must specify for a final solution that $t_2 \geq t_1$, *i.e.*,

$$f_{T_1,T_2}(t_1, t_2) = \begin{cases} \lambda^2 e^{-\lambda t_2}, & t_2 \geq t_1 \geq 0 \\ 0, & o.w. \end{cases}$$

(b) The support of (T_1, T_2) is $t_2 \geq t_1 \geq 0$ and is shown in Fig. 1.

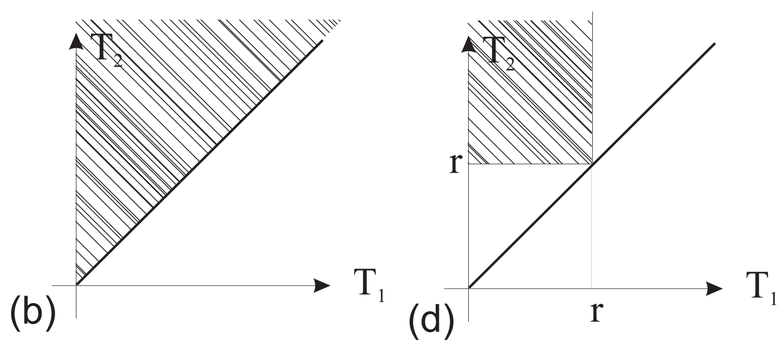

Figure 1: (b) Support of (T_1, T_2), and (d) event $\{T_1 < r\} \cap \{T_2 \geq r\}$ in Problem (6).

(c) See Fig. 1. For $r \geq 0$,

$$\begin{aligned} P[A] &= P[\{T_1 < r\} \cap \{T_2 \geq r\}] \\ &= \int_{t_1=0}^{r} \int_{t_2=r}^{\infty} f_{T_1,T_2}(t_1, t_2) dt_1 dt_2 \\ &= \lambda^2 \int_{t_1=0}^{r} \int_{t_2=r}^{\infty} e^{-\lambda t_2} dt_1 dt_2 \\ &= \lambda^2 \left(\int_{t_1=0}^{r} dt_1 \right) \left(\int_{t_2=r}^{\infty} e^{-\lambda t_2} dt_2 \right) \\ &= \lambda^2 r \frac{1}{\lambda} e^{-\lambda r} = \lambda r e^{-\lambda r} \end{aligned}$$

Equivalently, since I did not require that you integrate to find $P[A]$, you could have used the Poisson pmf to find the probability that $N(r) = 1$, *i.e.*, the number of arrivals at time r equals 1,

$$p_{N(r)}(1) = P[N(r) = 1] = \frac{(\lambda r)^1}{1!}e^{-\lambda r} = \begin{cases} \lambda r e^{-\lambda r}, & r > 0 \\ 0, & o.w. \end{cases}$$

(d) This solution requires starting with the joint pdf of T_1, T_2, that is, $f_{T_1,T_2}(t_1, t_2)$. Step 2, condition on the event A to find $f_{T_1,T_2|A}(t_1, t_2|A)$. Step 3, integrate out T_2 to find $f_{T_1|A}(t_1)$. This three step solution is longer than just using $f_{T_1}(t_1)$ and conditioning on A, because the event A depends on both T_1 and T_2. Below, you can see Step 3 (integrate t_2 from r to ∞), Step 2 ($\frac{f_{T_1,T_2}(t_1,t_2)}{P[A]}$ is the conditional pdf of T_1, T_2 given event A, and Step 1 (plug in the joint pdf from above):

$$\begin{aligned} f_{T_1|A}(t_1) &= \int_{t_2=r}^{\infty} \frac{f_{T_1,T_2}(t_1, t_2)}{P[A]} dt_2 = \int_{t_2=r}^{\infty} \frac{\lambda^2 e^{-\lambda t_2}}{\lambda r e^{-\lambda r}} dt_2 \\ &= \frac{\lambda}{r} e^{\lambda r} \int_{t_2=r}^{\infty} e^{-\lambda t_2} dt_2 = \frac{1}{r} e^{\lambda r} e^{-\lambda r} = \frac{1}{r} \\ &= \begin{cases} \frac{1}{r}, & 0 \le t_1 \le r \\ 0, & o.w. \end{cases} \end{aligned}$$

Thus t_1 is uniform between 0 and r! This means that given that there was exactly one arrival in the interval $[0, r]$, it was equally likely to be at any time in that interval.

7. Using $\lambda = 1/10$ minutes,

(a) Let $t = 0$ begin when Hacker A starts to break in, and T_1 be the time of the first system check by the admin. T_1 is Exponential, so from the CDF of the Exponential distribution,

$$P[A \text{ won't get caught}] = P[T_1 > 10] = 1 - (1 - e^{-10/10}) = e^{-1} \approx 0.368$$

Or, equivalently, let N be the number of police checks in the 10 minute period, which has a Poisson pmf:

$$P[A \text{ won't get caught}] = P[N = 0] = P_N(0) = e^{-10/10} = e^{-1}$$

(b) Here, let M be the number of police checks in the 20 minute period, which also has a Poisson pmf:

$$\begin{aligned} P[B \text{ won't get caught}] &= P[M < 2] = P_M(0) + P_M(1) \\ &= e^{-20/10} + \frac{(20/10)^1}{1!} e^{-20/10} = e^{-2} + 2e^{-2} \approx 0.406 \end{aligned}$$

(Hacker B has better chances of getting away with it!)

(a) From Theorem 5.13, $\mathbf{Y}$ has covariance matrix

$$\mathbf{C_Y} = \mathbf{Q}\mathbf{C_X}\mathbf{Q}' \tag{1}$$
$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{2}$$
$$= \begin{bmatrix} \sigma_1^2\cos^2\theta + \sigma_2^2\sin^2\theta & (\sigma_1^2 - \sigma_2^2)\sin\theta\cos\theta \\ (\sigma_1^2 - \sigma_2^2)\sin\theta\cos\theta & \sigma_1^2\sin^2\theta + \sigma_2^2\cos^2\theta \end{bmatrix}. \tag{3}$$

We conclude that Y_1 and Y_2 have covariance

$$\text{Cov}\,[Y_1, Y_2] = C_{\mathbf{Y}}(1,2) = (\sigma_1^2 - \sigma_2^2)\sin\theta\cos\theta. \tag{4}$$

Since Y_1 and Y_2 are jointly Gaussian, they are independent if and only if $\text{Cov}[Y_1, Y_2] = 0$. Thus, Y_1 and Y_2 are independent for all θ if and only if $\sigma_1^2 = \sigma_2^2$. In this case, when the joint PDF $f_{\mathbf{X}}(\mathbf{x})$ is symmetric in x_1 and x_2. In terms of polar coordinates, the PDF $f_{\mathbf{X}}(\mathbf{x}) = f_{X_1,X_2}(x_1, x_2)$ depends on $r = \sqrt{x_1^2 + x_2^2}$ but for a given r, is constant for all $\phi = \tan^{-1}(x_2/x_1)$. The transformation of $\mathbf{X}$ to $\mathbf{Y}$ is just a rotation of the coordinate system by θ preserves this circular symmetry.

(b) If $\sigma_2^2 > \sigma_1^2$, then Y_1 and Y_2 are independent if and only if $\sin\theta\cos\theta = 0$. This occurs in the following cases:

- $\theta = 0$: $Y_1 = X_1$ and $Y_2 = X_2$
- $\theta = \pi/2$: $Y_1 = -X_2$ and $Y_2 = -X_1$
- $\theta = \pi$: $Y_1 = -X_1$ and $Y_2 = -X_2$
- $\theta = -\pi/2$: $Y_1 = X_2$ and $Y_2 = X_1$

In all four cases, Y_1 and Y_2 are just relabeled versions, possibly with sign changes, of X_1 and X_2. In these cases, Y_1 and Y_2 are independent because X_1 and X_2 are independent. For other values of θ, each Y_i is a linear combination of both X_1 and X_2. This mixing results in correlation between Y_1 and Y_2.

Problem 5.7.7 Solution

The difficulty of this problem is overrated since its a pretty simple application of Problem 5.7.6. In particular,

$$\mathbf{Q} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\Bigg|_{\theta=45^\circ} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \tag{1}$$

Since $\mathbf{X} = \mathbf{Q}\mathbf{Y}$, we know from Theorem 5.16 that $\mathbf{X}$ is Gaussian with covariance matrix

$$\mathbf{C_X} = \mathbf{Q}\mathbf{C_Y}\mathbf{Q}' \tag{2}$$
$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{3}$$
$$= \frac{1}{2} \begin{bmatrix} 1+\rho & -(1-\rho) \\ 1+\rho & 1-\rho \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{4}$$
$$= \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}. \tag{5}$$

W_n both use X_{n-1} in their averaging, W_{n-1} and W_n are dependent. We can verify this observation by calculating the covariance of W_{n-1} and W_n. First, we observe that for all n,

$$E[W_n] = (E[X_n] + E[X_{n-1}])/2 = 30 \tag{1}$$

Next, we observe that W_{n-1} and W_n have covariance

$$\begin{aligned}\operatorname{Cov}[W_{n-1}, W_n] &= E[W_{n-1}W_n] - E[W_n]E[W_{n-1}] &(2)\\ &= \frac{1}{4}E[(X_{n-1} + X_{n-2})(X_n + X_{n-1})] - 900 &(3)\end{aligned}$$

We observe that for $n \neq m$, $E[X_nX_m] = E[X_n]E[X_m] = 900$ while

$$E\left[X_n^2\right] = \operatorname{Var}[X_n] + (E[X_n])^2 = 916 \tag{4}$$

Thus,

$$\operatorname{Cov}[W_{n-1}, W_n] = \frac{900 + 916 + 900 + 900}{4} - 900 = 4 \tag{5}$$

Since $\operatorname{Cov}[W_{n-1}, W_n] \neq 0$, W_n and W_{n-1} must be dependent.

Problem 10.4.3 Solution

The number Y_k of failures between successes $k-1$ and k is exactly $y \geq 0$ iff after success $k-1$, there are y failures followed by a success. Since the Bernoulli trials are independent, the probability of this event is $(1-p)^y p$. The complete PMF of Y_k is

$$P_{Y_k}(y) = \begin{cases} (1-p)^y p & y = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Since this argument is valid for all k including $k = 1$, we can conclude that $Y_1, Y_2, \ldots$ are identically distributed. Moreover, since the trials are independent, the failures between successes $k-1$ and k and the number of failures between successes $k'-1$ and k' are independent. Hence, $Y_1, Y_2, \ldots$ is an iid sequence.

Problem 10.5.1 Solution

This is a very straightforward problem. The Poisson process has rate $\lambda = 4$ calls per second. When t is measured in seconds, each $N(t)$ is a Poisson random variable with mean $4t$ and thus has PMF

$$P_{N(t)}(n) = \begin{cases} \frac{(4t)^n}{n!} e^{-4t} & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Using the general expression for the PMF, we can write down the answer for each part.

(a) $P_{N(1)}(0) = 4^0 e^{-4}/0! = e^{-4} \approx 0.0183$.

(b) $P_{N(1)}(4) = 4^4 e^{-4}/4! = 32e^{-4}/3 \approx 0.1954$.

(c) $P_{N(2)}(2) = 8^2 e^{-8}/2! = 32e^{-8} \approx 0.0107$.

Problem 10.5.2 Solution

Following the instructions given, we express each answer in terms of $N(m)$ which has PMF

$$P_{N(m)}(n) = \begin{cases} (6m)^n e^{-6m}/n! & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

(a) The probability of no queries in a one minute interval is $P_{N(1)}(0) = 6^0 e^{-6}/0! = 0.00248$.

(b) The probability of exactly 6 queries arriving in a one minute interval is $P_{N(1)}(6) = 6^6 e^{-6}/6! = 0.161$.

(c) The probability of exactly three queries arriving in a one-half minute interval is $P_{N(0.5)}(3) = 3^3 e^{-3}/3! = 0.224$.

Problem 10.5.3 Solution

Since there is always a backlog an the service times are iid exponential random variables, The time between service completions are a sequence of iid exponential random variables. that is, the service completions are a Poisson process. Since the expected service time is 30 minutes, the rate of the Poisson process is $\lambda = 1/30$ per minute. Since t hours equals $60t$ minutes, the expected number serviced is $\lambda(60t)$ or $2t$. Moreover, the number serviced in the first t hours has the Poisson PMF

$$P_{N(t)}(n) = \begin{cases} \frac{(2t)^n e^{-2t}}{n!} & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Problem 10.5.4 Solution

Since $D(t)$ is a Poisson process with rate 0.1 drops/day, the random variable $D(t)$ is a Poisson random variable with parameter $\alpha = 0.1t$. The PMF of $D(t)$. the number of drops after t days, is

$$P_{D(t)}(d) = \begin{cases} (0.1t)^d e^{-0.1t}/d! & d = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Problem 10.5.5 Solution

Note that it matters whether $t \geq 2$ minutes. If $t \leq 2$, then any customers that have arrived must still be in service. Since a Poisson number of arrivals occur during $(0, t]$,

$$P_{N(t)}(n) = \begin{cases} (\lambda t)^n e^{-\lambda t}/n! & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad (0 \leq t \leq 2) \quad (1)$$

For $t \geq 2$, the customers in service are precisely those customers that arrived in the interval $(t-2, t]$. The number of such customers has a Poisson PMF with mean $\lambda[t - (t-2)] = 2\lambda$. The resulting PMF of $N(t)$ is

$$P_{N(t)}(n) = \begin{cases} (2\lambda)^n e^{-2\lambda}/n! & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad (t \geq 2) \quad (2)$$

Problem 10.5.6 Solution

The time T between queries are independent exponential random variables with PDF

$$f_T(t) = \begin{cases} (1/8)e^{-t/8} & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

From the PDF, we can calculate for $t > 0$,

$$P[T \geq t] = \int_0^t f_T(t')\, dt' = e^{-t/8} \tag{2}$$

Using this formula, each question can be easily answered.

(a) $P[T \geq 4] = e^{-4/8} \approx 0.951$.

(b)

$$P[T \geq 13|T \geq 5] = \frac{P[T \geq 13, T \geq 5]}{P[T \geq 5]} \tag{3}$$

$$= \frac{P[T \geq 13]}{P[T \geq 5]} = \frac{e^{-13/8}}{e^{-5/8}} = e^{-1} \approx 0.368 \tag{4}$$

(c) Although the time betwen queries are independent exponential random variables, $N(t)$ is not exactly a Poisson random process because the first query occurs at time $t = 0$. Recall that in a Poisson process, the first arrival occurs some time after $t = 0$. However $N(t) - 1$ is a Poisson process of rate 8. Hence, for $n = 0, 1, 2, \ldots$,

$$P[N(t) - 1 = n] = (t/8)^n e^{-t/8}/n! \tag{5}$$

Thus, for $n = 1, 2, \ldots$, the PMF of $N(t)$ is

$$P_{N(t)}(n) = P[N(t) - 1 = n - 1] = (t/8)^{n-1} e^{-t/8}/(n-1)! \tag{6}$$

The complete expression of the PMF of $N(t)$ is

$$P_{N(t)}(n) = \begin{cases} (t/8)^{n-1} e^{-t/8}/(n-1)! & n = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Problem 10.5.7 Solution

This proof is just a simplified version of the proof given for Theorem 10.3. The first arrival occurs at time $X_1 > x \geq 0$ iff there are no arrivals in the interval $(0, x]$. Hence, for $x \geq 0$,

$$P[X_1 > x] = P[N(x) = 0] = (\lambda x)^0 e^{-\lambda x}/0! = e^{-\lambda x} \tag{1}$$

Since $P[X_1 \leq x] = 0$ for $x < 0$, the CDF of X_1 is the exponential CDF

$$F_{X_1}(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases} \tag{2}$$