ECE 5510 Fall 2009: Homework 8 Solutions

1. Y&G 10.8.1: See attached pages.

2. Y&G 10.8.3: See attached pages.

3. Y&G 10.8.4: See attached pages. Only $k \geq 0$ was required for part b. It was okay to leave it at line (9). Part d is a matter of opinion; an average of 0 degrees C would be a bad model for some parts of the world. Also, why should different days of January have different variance? This model has a particular variance as a function of day n, which you'd want to see if it matched the actual variance vs. day in January.

4. Y&G 10.10.1: $R_1(\tau)$ and $R_2(\tau)$ and $R_4(\tau)$ are valid autocorrelation functions. $R_3(\tau)$ is invalid because $R_3(0) = 0$ is not the maximum autocorrelation (the max is at $\tau = 10$), and it is not symmetric about $\tau = 0$. (True, as the Y&G solutions state, $R_1(\tau) = \delta(\tau)$ implies the power of the random process is infinite since $R_1(0) = \infty$, which is not physically realizable. However, $R_1(\tau)$ approximates the autocorrelation function for thermal noise, a random process that does exist and we use all the time as engineers. Since this is an engineering class, my answer is correct.)

5. Y&G 10.10.3: See attached pages.

6. (a) We know the means are zero, and that for $\tau = 1$, the covariance is 2 and the variances are 4 ($C_X(0)$ is the variance). You could have written the mean vector as $\mu_{\mathbf{X}} = [0, 0]^T$ and covariance matrix $C_{\mathbf{X}} = [4, 2; 2, 4]$ and then the joint pdf for vector $\mathbf{X} = [X(t), X(t+1)]^T$ could be written using the multivariate Gaussian pdf given in Definition 5.17 on page 229 of Y&G. Alternatively, you could have found that the correlation coefficient $\rho = 2/\sqrt{(4)(4)} = 1/2$, and used Definition 4.17 on page 191 of Y&G:

$$f_{X(t),X(t+1)}(x_1, x_2) = \frac{1}{2\pi(4)\sqrt{3/4}} \exp\left\{-\frac{1}{2(3/4)}\left[x_1^2/4 - x_1x_2/4 + x_2^2/4\right]\right\}$$

$$f_{X(t),X(t+1)}(x_1, x_2) = \frac{1}{4\pi\sqrt{3}} \exp\left\{-\frac{1}{6}\left[x_1^2 - x_1x_2 + x_2^2\right]\right\}$$

   (b) Now for $\tau = 3$, the covariance is 0. Thus $X(t)$ and $X(t+3)$ are independent, and we find the joint pdf by multiplying two marginal pdfs together:

$$f_{X(t),X(t+3)}(x_1, x_2) = \frac{1}{8\pi} \exp\left\{-\frac{1}{8}\left[x_1^2 + x_2^2\right]\right\}$$

In terms of matrices, $\mathbf{W} = \mathbf{A}\mathbf{X}$ where $\mathbf{A}$ is the lower triangular matrix

$$\mathbf{A} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & \cdots & \cdots & 1 \end{bmatrix}. \tag{4}$$

Since $E[\mathbf{W}] = \mathbf{A}E[\mathbf{X}] = \mathbf{0}$, it folows from Theorem 5.16 that

$$f_{\mathbf{W}}(\mathbf{w}) = \frac{1}{|\det(\mathbf{A})|} f_{\mathbf{X}}\left(\mathbf{A}^{-1}\mathbf{w}\right). \tag{5}$$

Since $\mathbf{A}$ is a lower triangular matrix, $\det(\mathbf{A}) = 1$, the product of its diagonal entries. In addition, reflecting the fact that each $X_n = W_n - W_{n-1}$,

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ 0 & -1 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^{-1}\mathbf{W} = \begin{bmatrix} W_1 \\ W_2 - W_1 \\ W_3 - W_2 \\ \vdots \\ W_k - W_{k-1} \end{bmatrix}. \tag{6}$$

Combining these facts with the observation that $f_{\mathbf{X}}(\mathbf{x}) = \prod_{n=1}^{k} f_{X_n}(x_n)$, we can write

$$f_{\mathbf{W}}(\mathbf{w}) = f_{\mathbf{X}}\left(\mathbf{A}^{-1}\mathbf{w}\right) = \prod_{n=1}^{k} f_{X_n}(w_n - w_{n-1}), \tag{7}$$

which completes the missing steps in the proof of Theorem 10.8.

Problem 10.8.1 Solution

The discrete time autocovariance function is

$$C_X[m,k] = E[(X_m - \mu_X)(X_{m+k} - \mu_X)] \tag{1}$$

for $k = 0$, $C_X[m,0] = \text{Var}[X_m] = \sigma_X^2$. For $k \neq 0$, X_m and X_{m+k} are independent so that

$$C_X[m,k] = E[(X_m - \mu_X)]\,E[(X_{m+k} - \mu_X)] = 0 \tag{2}$$

Thus the autocovariance of X_n is

$$C_X[m,k] = \begin{cases} \sigma_X^2 & k = 0 \\ 0 & k \neq 0 \end{cases} \tag{3}$$

Problem 10.8.2 Solution

Recall that $X(t) = t - W$ where $E[W] = 1$ and $E[W^2] = 2$.

(a) The mean is $\mu_X(t) = E[t - W] = t - E[W] = t - 1$.

(b) The autocovariance is

$$\begin{aligned} C_X(t,\tau) &= E[X(t)X(t+\tau)] - \mu_X(t)\mu_X(t+\tau) && (1) \\ &= E[(t-W)(t+\tau-W)] - (t-1)(t+\tau-1) && (2) \\ &= t(t+\tau) - E[(2t+\tau)W] + E\left[W^2\right] - t(t+\tau) + 2t + \tau - 1 && (3) \\ &= -(2t+\tau)E[W] + 2 + 2t + \tau - 1 && (4) \\ &= 1 && (5) \end{aligned}$$

Problem 10.8.3 Solution

In this problem, the daily temperature process results from

$$C_n = 16\left[1 - \cos\frac{2\pi n}{365}\right] + 4X_n \tag{1}$$

where X_n is an iid random sequence of $N[0,1]$ random variables. The hardest part of this problem is distinguishing between the process C_n and the covariance function $C_C[k]$.

(a) The expected value of the process is

$$E[C_n] = 16E\left[1 - \cos\frac{2\pi n}{365}\right] + 4E[X_n] = 16\left[1 - \cos\frac{2\pi n}{365}\right] \tag{2}$$

(b) The autocovariance of C_n is

$$C_C[m,k] = E\left[\left(C_m - 16\left[1 - \cos\frac{2\pi m}{365}\right]\right)\left(C_{m+k} - 16\left[1 - \cos\frac{2\pi(m+k)}{365}\right]\right)\right] \tag{3}$$

$$= 16E[X_m X_{m+k}] = \begin{cases} 16 & k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(c) A model of this type may be able to capture the mean and variance of the daily temperature. However, one reason this model is overly simple is because day to day temperatures are uncorrelated. A more realistic model might incorporate the effects of “heat waves” or “cold spells” through correlated daily temperatures.

Problem 10.8.4 Solution

By repeated application of the recursion $C_n = C_{n-1}/2 + 4X_n$, we obtain

$$C_n = \frac{C_{n-2}}{4} + 4\left[\frac{X_{n-1}}{2} + X_n\right] \tag{1}$$

$$= \frac{C_{n-3}}{8} + 4\left[\frac{X_{n-2}}{4} + \frac{X_{n-1}}{2} + X_n\right] \tag{2}$$

$$\vdots \tag{3}$$

$$= \frac{C_0}{2^n} + 4\left[\frac{X_1}{2^{n-1}} + \frac{X_2}{2^{n-2}} + \cdots + X_n\right] = \frac{C_0}{2^n} + 4\sum_{i=1}^{n}\frac{X_i}{2^{n-i}} \tag{4}$$

(a) Since $C_0, X_1, X_2, \ldots$ all have zero mean,

$$E[C_n] = \frac{E[C_0]}{2^n} + 4\sum_{i=1}^{n}\frac{E[X_i]}{2^{n-i}} = 0 \tag{5}$$

(b) The autocovariance is

$$C_C[m,k] = E\left[\left(\frac{C_0}{2^n} + 4\sum_{i=1}^{n}\frac{X_i}{2^{n-i}}\right)\left(\frac{C_0}{2^m + k} + 4\sum_{j=1}^{m+k}\frac{X_j}{2^{m+k-j}}\right)\right] \tag{6}$$

Since $C_0, X_1, X_2, \ldots$ are independent (and zero mean), $E[C_0 X_i] = 0$. This implies

$$C_C[m,k] = \frac{E\left[C_0^2\right]}{2^{2m+k}} + 16 \sum_{i=1}^{m} \sum_{j=1}^{m+k} \frac{E\left[X_i X_j\right]}{2^{m-i} 2^{m+k-j}} \tag{7}$$

For $i \neq j$, $E[X_i X_j] = 0$ so that only the $i = j$ terms make any contribution to the double sum. However, at this point, we must consider the cases $k \geq 0$ and $k < 0$ separately. Since each X_i has variance 1, the autocovariance for $k \geq 0$ is

$$C_C[m,k] = \frac{1}{2^{2m+k}} + 16 \sum_{i=1}^{m} \frac{1}{2^{2m+k-2i}} \tag{8}$$

$$= \frac{1}{2^{2m+k}} + \frac{16}{2^k} \sum_{i=1}^{m} (1/4)^{m-i} \tag{9}$$

$$= \frac{1}{2^{2m+k}} + \frac{16}{2^k} \frac{1 - (1/4)^m}{3/4} \tag{10}$$

For $k < 0$, we can write

$$C_C[m,k] = \frac{E\left[C_0^2\right]}{2^{2m+k}} + 16 \sum_{i=1}^{m} \sum_{j=1}^{m+k} \frac{E\left[X_i X_j\right]}{2^{m-i} 2^{m+k-j}} \tag{11}$$

$$= \frac{1}{2^{2m+k}} + 16 \sum_{i=1}^{m+k} \frac{1}{2^{2m+k-2i}} \tag{12}$$

$$= \frac{1}{2^{2m+k}} + \frac{16}{2^{-k}} \sum_{i=1}^{m+k} (1/4)^{m+k-i} \tag{13}$$

$$= \frac{1}{2^{2m+k}} + \frac{16}{2^k} \frac{1 - (1/4)^{m+k}}{3/4} \tag{14}$$

A general expression that's valid for all m and k is

$$C_C[m,k] = \frac{1}{2^{2m+k}} + \frac{16}{2^{|k|}} \frac{1 - (1/4)^{\min(m,m+k)}}{3/4} \tag{15}$$

(c) Since $E[C_i] = 0$ for all i, our model has a mean daily temperature of zero degrees Celsius for the entire year. This is not a reasonable model for a year.

(d) For the month of January, a mean temperature of zero degrees Celsius seems quite reasonable. we can calculate the variance of C_n by evaluating the covariance at $n = m$. This yields

$$\text{Var}[C_n] = \frac{1}{4^n} + \frac{16}{4^n} \frac{4(4^n - 1)}{3} \tag{16}$$

Note that the variance is upper bounded by

$$\text{Var}[C_n] \leq 64/3 \tag{17}$$

Hence the daily temperature has a standard deviation of $8/\sqrt{3} \approx 4.6$ degrees. Without actual evidence of daily temperatures in January, this model is more difficult to discredit.

Problem 10.10.1 Solution

The autocorrelation function $R_X(\tau) = \delta(\tau)$ is mathematically valid in the sense that it meets the conditions required in Theorem 10.12. That is,

$$R_X(\tau) = \delta(\tau) \geq 0 \tag{1}$$
$$R_X(\tau) = \delta(\tau) = \delta(-\tau) = R_X(-\tau) \tag{2}$$
$$R_X(\tau) \leq R_X(0) = \delta(0) \tag{3}$$

However, for a process $X(t)$ with the autocorrelation $R_X(\tau) = \delta(\tau)$, Definition 10.16 says that the average power of the process is

$$E\left[X^2(t)\right] = R_X(0) = \delta(0) = \infty \tag{4}$$

Processes with infinite average power cannot exist in practice.

Problem 10.10.2 Solution

Since $Y(t) = A + X(t)$, the mean of $Y(t)$ is

$$E[Y(t)] = E[A] + E[X(t)] = E[A] + \mu_X \tag{1}$$

The autocorrelation of $Y(t)$ is

$$\begin{aligned} R_Y(t,\tau) &= E[(A + X(t))(A + X(t+\tau))] && (2)\\ &= E\left[A^2\right] + E[A]\,E[X(t)] + AE[X(t+\tau)] + E[X(t)X(t+\tau)] && (3)\\ &= E\left[A^2\right] + 2E[A]\,\mu_X + R_X(\tau) && (4) \end{aligned}$$

We see that neither $E[Y(t)]$ nor $R_Y(t,\tau)$ depend on t. Thus $Y(t)$ is a wide sense stationary process.

Problem 10.10.3 Solution

In this problem, we find the autocorrelation $R_W(t,\tau)$ when

$$W(t) = X\cos 2\pi f_0 t + Y\sin 2\pi f_0 t, \tag{1}$$

and X and Y are uncorrelated random variables with $E[X] = E[Y] = 0$.

We start by writing

$$\begin{aligned} R_W(t,\tau) &= E[W(t)W(t+\tau)] && (2)\\ &= E[(X\cos 2\pi f_0 t + Y\sin 2\pi f_0 t)(X\cos 2\pi f_0(t+\tau) + Y\sin 2\pi f_0(t+\tau))]\,. && (3) \end{aligned}$$

Since X and Y are uncorrelated, $E[XY] = E[X]E[Y] = 0$. Thus, when we expand $E[W(t)W(t+\tau)]$ and take the expectation, all of the XY cross terms will be zero. This implies

$$R_W(t,\tau) = E\left[X^2\right]\cos 2\pi f_0 t\cos 2\pi f_0(t+\tau) + E\left[Y^2\right]\sin 2\pi f_0 t\sin 2\pi f_0(t+\tau) \tag{4}$$

Since $E[X] = E[Y] = 0$,

$$E\left[X^2\right] = \text{Var}[X] - (E[X])^2 = \sigma^2, \qquad E\left[Y^2\right] = \text{Var}[Y] - (E[Y])^2 = \sigma^2. \tag{5}$$

In addition, from Math Fact B.2, we use the formulas

$$\cos A\cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right] \tag{6}$$
$$\sin A\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] \tag{7}$$

to write

$$R_W(t,\tau) = \frac{\sigma^2}{2}\left(\cos 2\pi f_0\tau + \cos 2\pi f_0(2t+\tau)\right) + \frac{\sigma^2}{2}\left(\cos 2\pi f_0\tau - \cos 2\pi f_0(2t+\tau)\right) \tag{8}$$

$$= \sigma^2 \cos 2\pi f_0 \tau \tag{9}$$

Thus $R_W(t,\tau) = R_W(\tau)$. Since

$$E[W(t)] = E[X]\cos 2\pi f_0 t + E[Y]\sin 2\pi f_0 t = 0, \tag{10}$$

we can conclude that $W(t)$ is a wide sense stationary process. However, we note that if $E[X^2] \neq E[Y^2]$, then the $\cos 2\pi f_0(2t+\tau)$ terms in $R_W(t,\tau)$ would not cancel and $W(t)$ would not be wide sense stationary.

Problem 10.10.4 Solution

(a) In the problem statement, we are told that $X(t)$ has average power equal to 1. By Definition 10.16, the average power of $X(t)$ is $E[X^2(t)] = 1$.

(b) Since Θ has a uniform PDF over $[0, 2\pi]$,

$$f_\Theta(\theta) = \begin{cases} 1/(2\pi) & 0 \leq \theta \leq 2\pi \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The expected value of the random phase cosine is

$$E[\cos(2\pi f_c t + \Theta)] = \int_{-\infty}^{\infty} \cos(2\pi f_c t + \theta) f_\Theta(\theta)\, d\theta \tag{2}$$

$$= \int_0^{2\pi} \cos(2\pi f_c t + \theta)\frac{1}{2\pi} d\theta \tag{3}$$

$$= \frac{1}{2\pi}\sin(2\pi f_c t + \theta)\Big|_0^{2\pi} \tag{4}$$

$$= \frac{1}{2\pi}\left(\sin(2\pi f_c t + 2\pi) - \sin(2\pi f_c t)\right) = 0 \tag{5}$$

(c) Since $X(t)$ and Θ are independent,

$$E[Y(t)] = E[X(t)\cos(2\pi f_c t + \Theta)] = E[X(t)]\, E[\cos(2\pi f_c t + \Theta)] = 0 \tag{6}$$

Note that the mean of $Y(t)$ is zero no matter what the mean of $X(t)$ since the random phase cosine has zero mean.

(d) Independence of $X(t)$ and Θ results in the average power of $Y(t)$ being

$$E\left[Y^2(t)\right] = E\left[X^2(t)\cos^2(2\pi f_c t + \Theta)\right] \tag{7}$$

$$= E\left[X^2(t)\right] E\left[\cos^2(2\pi f_c t + \Theta)\right] \tag{8}$$

$$= E\left[\cos^2(2\pi f_c t + \Theta)\right] \tag{9}$$

Note that we have used the fact from part (a) that $X(t)$ has unity average power. To finish the problem, we use the trigonometric identity $\cos^2\phi = (1 + \cos 2\phi)/2$. This yields

$$E\left[Y^2(t)\right] = E\left[\frac{1}{2}\left(1 + \cos(2\pi(2f_c)t + \Theta)\right)\right] = 1/2 \tag{10}$$

Note that $E[\cos(2\pi(2f_c)t + \Theta)] = 0$ by the argument given in part (b) with $2f_c$ replacing f_c.